EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE - 2014/2015

FIRST SEMESTER (Sep./Oct., 2016)

103 - VECTOR ALGEBRA & CLASSICAL MECHANICS I (REPEAT)

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ver all questions — Time : Three hours

(a) For any three vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$, prove that the identity

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}.$$

Let $\underline{l}$, $\underline{m}$ and $\underline{n}$ be three non zero and non co-planer vectors such that any two of them are not parallel. By considering the vector product $(\underline{r} \wedge \underline{l}) \wedge (\underline{m} \wedge \underline{n})$, prove that any vector $\underline{r}$ can be expressed in the form

$$\underline{r} = (\underline{r} \cdot \underline{\alpha})\underline{l} + (\underline{r} \cdot \underline{\beta})\underline{m} + (\underline{r} \cdot \underline{\gamma})\underline{n}.$$

Hence find the vectors $\underline{\alpha}$, $\underline{\beta}$ and $\underline{\gamma}$ in terms of $\underline{l}$, $\underline{m}$ and $\underline{n}$.

(b) Find the equation of the plane passing through three given terminal points of $\underline{a}$, $\underline{b}$ and $\underline{c}$.

(c) Find the volume of the parallelepiped whose edges are represented by $(2, -3, 4)$, $(1, 2, -1)$ and $(3, -1, 2)$.

2. Define the following terms:

- gradient of a scalar field;
- divergence of a vector field.

(a) Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$, $r = |\underline{r}|$ and $\underline{a}$ be a constant vector. Find $\text{div}(r^n\underline{r})$, where n is a constant. Show that

$$\text{grad}\left(\frac{\underline{a}\cdot\underline{r}}{r^3}\right) = \frac{\underline{a}}{r^3} + 3\frac{(\underline{a}\cdot\underline{r})}{r^5}\ \underline{r}.$$

(b) Find the directional derivative of $\phi = 2x^3 - 3yz$ at the point $(2, 1, 3)$ in the direction parallel to the line whose direction cosines are proportional to $(2, 1, 2)$.

(c) Determine the constant 'a' so that the vector

$$\underline{F} = (x + 3y)\underline{i} + (y - 2z)\underline{j} + (x + az)\underline{k}$$

is solenoidal.

3. (a) Let $O = (0, 0, 0)$, $A = (1, 0, 0)$, $B = (1, 2, 0)$ and $C = (1, 2, 3)$. By considering the straight line path OA, AB, BC, find the line integral $\int_\gamma \underline{F}\cdot d\underline{r}$, where γ is a path from O to C and $\underline{F} = (2y + 3)\ \underline{i} + xz\ \underline{j} + (yz - x)\ \underline{k}$.

(b) State the Divergence theorem. 6

Verify the Divergence theorem for $\underline{F} = 4xz\underline{i} - y^2\underline{j} + yz\underline{k}$ and S is the surface of the cube bounded by the planes $x = 0,\ x = 1,\ y = 0,\ y = 1,\ z = 0$ and $z = 1$.

4. (a) Prove that the radial and transverse component of the acceleration of a particle in terms of the polar co-ordinates (r, θ) are

$$\ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \ \text{respectively.}$$

(b) A particle of mass m rests on a smooth horizontal table attached through a fixe point on the table by a light elastic string of modules mg and unstretched lengt 'a'. Initially the string is just taut and the particle is projected along the tab in a direction perpendicular to the line of the string with velocity $\sqrt{\frac{4ag}{3}}$. Pro that if r is the distance of the particle from the fixed point at time t then

$$\frac{d^2r}{dt^2} = \frac{4ga^3}{3r^3} - \frac{g(r-a)}{a}.$$

Prove also that the string will extend until its length is $2a$ and that the velocity of the particle is half of its initial velocity.

(a) A particle moves in a plane with the velocity v and the tangent to the path of the particle makes an angle ψ with a fixed line in the plane. Prove that the components of acceleration of the particle along the tangent and perpendicular to it are $\frac{dv}{dt}$ and $v\frac{d\psi}{dt}$ respectively.

(b) A body attached to a parachute is released from an aeroplane which is moving horizontally with velocity v_0. The parachute exerts a drag opposing motion which is k times the weight of the body , where k is a constant. Neglecting the air resistance to the motion of the body, prove that if v is the velocity of the body when its path is inclined an angle ψ to the horizontal, then

$$v = \frac{v_0 \sec\psi}{(\sec\psi + \tan\psi)^k}.$$

Prove that if $k = 1$, the body cannot have a vertical component of velocity greater than $\frac{v_0}{2}$.

(a) State the angular momentum principle for motion of a particle.

(b) A right circular cone with a semi vertical angle α is fixed with its axis vertical and vertex downwards. A particle of mass m is held at the point A on the smooth inner surface of the cone at a distance $'a'$ from the axis of revolution. The particle is projected perpendicular to OA with velocity $'u'$, where O is the vertex of the cone. Show that the particle rises above the level of A if $u^2 > ag\cot\alpha$ and greatest reaction between the particle and the surface is

$$mg\left(\sin\alpha + \frac{u^2}{ag}\cos\alpha\right).$$