EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE - 2015/2016

FIRST SEMESTER (July,/Oct., 2017)

AM 103 - VECTOR ALGEBRA & CLASSICAL MECHANICS I (REPEAT)

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nswer all questions   Time : Three hours

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1. (a) For any three vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, prove that

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}.$$

Hence show that

$$(\underline{a} \wedge \underline{b}) \cdot (\underline{c} \wedge \underline{d}) = (\underline{a} \cdot \underline{c})(\underline{b} \cdot \underline{d}) - (\underline{a} \cdot \underline{d})(\underline{b} \cdot \underline{c}).$$

(b) Find an equation for the plane passing through three points whose position vectors are given by $(2, -1, 1)$, $(3, 2, -1)$ and $(-1, 3, 2)$.

(c) Find the vector $\underline{x}$ and the scalar λ which satisfy the equations

$$\underline{a} \wedge \underline{x} = \underline{b} + \lambda \underline{a}, \qquad \underline{a} \cdot \underline{x} = 2,$$

where $\underline{a} = \underline{i} + 2\underline{j} - \underline{k}$ and $\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$.

2. Define the following terms:

- gradient of a scalar field;
- divergence of a vector field.

(a) Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$, $r = |\underline{r}|$ and $\underline{a}$ be a constant vector. Find $\text{div}(r^n \underline{r})$, where n is a constant. Show that

$$\text{grad}\left(\frac{\underline{a} \cdot \underline{r}}{r^3}\right) = \frac{\underline{a}}{r^3} + 3\frac{(\underline{a} \cdot \underline{r})}{r^5}\ \underline{r}.$$

(b) Find the directional derivative of $\phi = 2x^3 - 3yz$ at the point $(2, 1, 3)$ in the direction parallel to the line whose direction cosines are proportional to $(2, 1, 2)$.

(c) Determine the constant 'a' so that the vector

$$\underline{F} = (x + 3y)\underline{i} + (y - 2z)\underline{j} + (x + az)\underline{k}\ .$$

is solenoidal.

3. (a) Define the following terms:

- *a conservative vector field;*
- *solenoidal vector field.*

Show that $\underline{F} = (2x - y)\underline{i} + (2yz^2 - x)\underline{j} + (2y^2z - z)\underline{k}$ is conservative but not solenoidal.

(b) Let $O = (0, 0, 0)$, $A = (1, 0, 0)$, $B = (1, 2, 0)$ and $C = (1, 2, 3)$. By considering the straight line path OA, AB, BC, find the line integral $\int_\gamma \underline{F} \cdot d\underline{r}$, where γ is a path from O to C.

(c) State the Divergence theorem, and use it to evaluate $\int\int_S \underline{F} \cdot \underline{n}\, dS$, where $\underline{F} = 4xz\underline{i} - y^2\underline{j} + yz\underline{k}$ and S is the surface of the cube bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 1$.

4. Prove that the radial and transverse component of the acceleration of a particle in terms of the polar co-ordinates (r, θ) are

$$\ddot{r} - r\dot{\theta}^2 \quad \text{and} \quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \quad \text{respectively.}$$

A light inextensible string of length $2a$ passes through a smooth ring at a point O, on a smooth horizontal table and two particles, each of mass m, attached to it's ends A and B. Initially the particles lie on the table with $OA = OB = a$ and AOB a straight line, the particle A is given a velocity u in a direction perpendicular to OA. Prove that if r and θ are the polar co-ordinates of A at a time t with respect to the origin, then

i. $2\,\dfrac{d^2r}{dt^2} - \dfrac{a^2u^2}{r^3} = 0,$

ii. $2r\,\dfrac{dr}{dt} = u\,\sqrt{2(r^2 - a^2)},$

iii. $r^2 = a^2 + \dfrac{1}{2}u^2t^2.$

Find the velocity of A at the instant when B reaches the origin at O.

5. A particle moves in a plane with the velocity v and the tangent to the path of the particle makes an angle ψ with a fixed line in the plane. Prove that the components of acceleration of the particle along the tangent and perpendicular to it are $\dfrac{dv}{dt}$ and $v\dfrac{d\psi}{dt}$ respectively.

A body attached to a parachute is released from an aeroplane which is moving horizontally with velocity v_0. The parachute exerts a drag opposing motion which is k times the weight of the body , where k is a constant. Neglecting the air resistance to the motion of the body, prove that if v is the velocity of the body when its path is inclined an angle ψ to the horizontal, then

$$v = \frac{v_0 \sec\psi}{(\sec\psi + \tan\psi)^k}.$$

Prove that if $k = 1$, the body cannot have a vertical component of velocity greater than $\dfrac{v_0}{2}$.

6. A rocket is fired upwards. Matter is ejected with constant relative velocity gT, at a constant rate $\frac{2M}{T}$. Initially the mass of the rocket is $2M$, half of this is available for ejection. Neglecting air resistance and variation in gravitational attraction, show that the greatest speed of the rocket is attained when the mass of the rocket is reduced to M and this speed is

$$gT\left(\ln 2 - \frac{1}{2}\right).$$

Show also that the rocket will reach the greatest height given by

$$\frac{1}{2}gT^2(1-\ln 2)^2.$$