



**EASTERN UNIVERSITY, SRI LANKA**  
**DEPARTMENT OF MATHEMATICS**  
**FIRST YEAR EXAMINATION IN SCIENCE - 2016/2017**  
**SECOND SEMESTER - (MARCH, 2019)**  
**AM 104 - DIFFERENTIAL EQUATIONS**  
**AND**  
**FOURIER SERIES**  
**(REPEAT)**

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Answer All Questions

Time Allowed: 3 Hours

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- Q1. (a) State the necessary and sufficient condition for the differential equation (DE)

$$M(x, y) dx + N(x, y) dy = 0 \quad (1)$$

to be exact.

[10 Marks]

- (b) Verify that the condition for an exact differential equation (DE) is satisfied by

$$[M(x, y)dx + N(x, y)dy]e^{\int f(x)dx} = 0$$

if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} + Nf(x).$$

[20 Marks]

Hence show that an integrating factor can always be found for the DE (1) if

$$\frac{1}{N} \left( \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \right)$$

is a function of  $x$  only.

[10 Marks]

Hence using this method solve the following DE

$$(y^2 + 3xy + 2y)dx + (xy + x^2 + x)dy = 0.$$

[30 Marks]

- (c) If,  $\tan x$ , is a particular solution of the following nonlinear Riccati equation

$$\frac{dy}{dx} - y^2 - 1 = 0, \quad (2)$$

then obtain the general solution of the equation (2).

[30 Marks]

- Q2. (a) If  $F(D) = \sum_{i=0}^n p_i D^i$ , where  $D \equiv \frac{d}{dx}$  and  $p_i, i = 1, \dots, n$ , are constants with  $p_0 \neq 0$ , prove the following formulas:

(i)  $\frac{1}{F(D)} e^{\alpha x} = \frac{1}{F(\alpha)} e^{\alpha x}$ , where  $\alpha$  is a constant and  $F(\alpha) \neq 0$ ;

(ii)  $\frac{1}{F(D)} e^{\alpha x} V = e^{\alpha x} \frac{1}{F(D + \alpha)} V$ , where  $V$  is a function of  $x$ .

[40 Marks]

- (b) Find the general solution of the following differential equations by using the results in (a):

(i)  $(D^4 - 2D^2 + 1)y = 40 \cosh x$ ;

(ii)  $(D^2 - 6D + 13)y = 8e^{3x} \sin 2x$ .

[60 Marks]

- Q3. (a) Let  $x = e^t$ . Show that

$$x \frac{d}{dx} \equiv \mathcal{D}, \quad x^2 \frac{d^2}{dx^2} \equiv \mathcal{D}^2 - \mathcal{D},$$

and

$$x^3 \frac{d^3}{dx^3} \equiv \mathcal{D}(\mathcal{D} - 1)(\mathcal{D} - 2),$$

where  $\mathcal{D} \equiv \frac{d}{dt}$ .

[20 Marks]

Use the above results to find the general solution of the following Cauchy-Euler differential equation

$$(x^3 D^3 + 3x^2 D^2 + xD + 8)y = 65 \cos(\log x),$$

where  $D \equiv \frac{d}{dx}$ .

[40 Marks]

(b) Define what is meant by *orthogonal trajectories of curves*.

[10 Marks]

Find the orthogonal trajectories of the family of curves in polar coordinates

$$r = \frac{a\theta}{1 + \theta},$$

where  $a$  is a constant.

[30 Marks]

Q4. (a) Define what is meant by the point, ' $x = x_0$ ', being

- (i) an *ordinary* ;
  - (ii) a *singular*;
  - (iii) a *regular singular*
- point of the DE

$$y'' + p(x)y' + q(x)y = 0,$$

where the prime denotes differentiation with respect to  $x$ , and  $p(x)$  and  $q(x)$  are rational functions.

[30 Marks]

(b) (i) Find the regular singular point(s) of the DE

$$4xy'' + 2y' - 7y = 0. \quad (3)$$

(ii) Use the method of Frobenius to find the general solution of the equation (3).

[70 Marks]

Q5. (a) Solve the following system of DEs:

$$(i) \frac{dx}{1} = \frac{dy}{-2} = \frac{dz}{3x^2 \sin(y + 2x)};$$

$$(ii) \frac{dx}{x^2 + y^2 - yz} = \frac{dy}{xz - x^2 - y^2} = \frac{dz}{z(x - y)}.$$

[30 Marks]

(b) Write down the condition of integrability of the total differential equation

$$P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz = 0.$$

[5 Marks]

Hence solve the following equation

$$(yz + xyz) dx + (xz + xyz) dy + (xy + xyz) dz = 0. \quad (4)$$

(You may use the integrating factor  $\mu = 1/(xyz)$  for equation (4).)

[15 Marks]

(c) Find the general solution of the following linear first-order partial differential equations:

(i)  $x^2p + y^2q = -z^2$ ;

(ii)  $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$ .

[30 Marks]

(d) Apply Charpit's method or otherwise to find the complete and the singular solution of the following non-linear first-order partial differential equation

$$2xz - px^2 - 2qxy + pq = 0.$$

Here,  $p = \frac{\partial z}{\partial x}$  and  $q = \frac{\partial z}{\partial y}$ .

[20 Marks]

Q6. (a) Utilize the Fourier series of the function

$$f(x) = |x|, \quad -1 \leq x \leq 1,$$

to show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos n\pi x = \frac{(2|x| - 1)\pi^2}{4}.$$

[20 Marks]

Hence deduce that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

[20 Marks]

(b) Use Fourier transform to solve the one-dimensional heat equation

$$\frac{\partial U}{\partial t} - 2 \frac{\partial^2 U}{\partial x^2} = 0,$$

subject to the boundary conditions

$$U(0, t) = 0, \quad U(x, 0) = e^{-x}, \quad x > 0$$

and  $U(x, t)$  is bounded where  $x > 0$  and  $t > 0$ .

[40 Marks]

(c) (i) Define the *gamma-function*  $\Gamma(x)$  and *beta-function*  $B(m, n)$ , where  $m, n$  are positive integers.

(ii) Evaluate the integral

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin \theta} d\theta.$$

(You may use the following results without proof

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.)$$

[20 Marks]

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