EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE - 2015/2016

FIRST SEMESTER (July/Aug., 2017)

103 - VECTOR ALGEBRA & CLASSICAL MECHANICS I

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ver all questions   Time : Three hours

(a) Prove that the diagonals of a parallelogram bisect each other.

(b) Find an equation for the plane perpendicular to the vector $2\underline{i} + 3\underline{j} + 6\underline{k}$ and passing through the terminal point of the vector $\underline{i} + 5\underline{j} + 3\underline{k}$.

(c) The vectors $\underline{\alpha}$, $\underline{\beta}$, $\underline{\gamma}$ are defined in terms of vectors $\underline{a}$, $\underline{b}$, $\underline{c}$ by

$$\underline{\alpha} = \left(\frac{\underline{b} \wedge \underline{c}}{V}\right), \quad \underline{\beta} = \left(\frac{\underline{c} \wedge \underline{a}}{V}\right), \quad \underline{\gamma} = \left(\frac{\underline{a} \wedge \underline{b}}{V}\right),$$

where $V = \underline{a} \cdot (\underline{b} \wedge \underline{c}) \neq 0$.

Show that

i. $\underline{\alpha} \cdot (\underline{\beta} \wedge \underline{\gamma}) = \dfrac{1}{V}$;

ii. any vector $\underline{r}$ can be expressed in the form $\underline{r} = (\underline{r} \cdot \underline{\alpha})\, \underline{a} + (\underline{r} \cdot \underline{\beta})\, \underline{b} + (\underline{r} \cdot \underline{\gamma})\, \underline{c}$.

2. (a) Define the following terms:

i. gradient of a scalar field ϕ ;

ii. curl of a vector field $\underline{F}$.

Prove that

$$\text{curl}(\phi \underline{F}) = \phi \ \text{curl}\underline{F} + \text{grad}\phi \wedge \underline{F}.$$

(b) Explain what is meant by "conservative vector field".

Show that

$$\underline{F} = (4xy - 3x^2z^2)\underline{i} + 2x^2\underline{j} - 2x^3z\underline{k}$$

is conservative vector field. Find the scalar potential function ϕ such $\underline{F} = \nabla\phi$.

(c) Let $\underline{a}$ be a non-zero constant vector and $\underline{r}$ be a position vector of a poin $r = |\underline{r}|$, find $\nabla\left(\frac{\underline{a}\cdot\underline{r}}{r^5}\right)$.

Hence show that

$$\nabla \wedge \left[\left(\frac{\underline{a}\cdot\underline{r}}{r^5}\right)\underline{r}\right] = \frac{\underline{a}\cdot\underline{r}}{r^5}.$$

3. (a) If $\underline{F} = y\ \underline{i} + (x - 2xz)\ \underline{j} - xy\ \underline{k}$, evaluate $\int\int_S (\nabla \wedge \underline{F}) \cdot \underline{n}\ dS$, where S is surface of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy plane.

(b) State the Green's Theorem in the plane.

Verify the Green's Theorem in the plane for

$$\oint_C (3x^2 - 8y^2)\ dx + (4y - 6xy)\ dy,$$

where C is the closed curve of the region bounded by $y = x^2$ and $y^2 = x$.

A particle A on a smooth table is attached by a string passing through a small hole in the table and carries a particle B of equal mass hanging vertically. The particle A is projected along the table at right angle to the string with velocity $\sqrt{2gh}$ when at a distance $'a'$ from the hole. Here g is the gravitational acceleration and h is a constant. If r is the distance of the particle A from the hole at time t, prove the following:

i. $\left(\frac{dr}{dt}\right)^2 = gh\left(1-\frac{a^2}{r^2}\right) + g(a-r);$

ii. the particle B will be pulled up to the hole if the total length of the string is less than $\frac{h}{2}+\sqrt{ah+\frac{h^2}{4}};$

iii. the tension of the string is $\frac{mg}{2}\left(1+\frac{2a^2h}{r^3}\right)$, where m is the mass of each particle.

A particle moves in a plane with the velocity v and the tangent to the path of the particle makes an angle ψ with a fixed line in the plane. Prove that the components of acceleration of the particle along the tangent and perpendicular to it are $\frac{dv}{dt}$ and $v\frac{d\psi}{dt}$ respectively.

A body attached to a parachute is released from an aeroplane which is moving horizontally with velocity v_0. The parachute exerts a drag opposing motion which is k times the weight of the body , where k is a constant. Neglecting the air resistance to the motion of the body, prove that if v is the velocity of the body when its path is inclined an angle ψ to the horizontal, then

$$v = \frac{v_0 \sec\psi}{(\sec\psi + \tan\psi)^k}\ .$$

Hence prove that if $k = 1$, the body cannot have a vertical component of velocity greater than $\frac{v_0}{2}$.

6. State the angular momentum principle for motion of a particle.

A particle is projected horizontally along the inner surface of a smooth cone, wh axis is vertical and vertex upwards. Find the pressure at any point in terms of depth below the vertex. Show that the particle will leave the cone at the depth be the vertex given by

$$\left(\frac{V^2h^2}{g\tan^2\alpha}\right)^{1/3},$$

where h is the initial depth, V is the initial velocity and α is the semi angle of cone.