EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE -2009/2010

FIRST SEMESTER (June/July, 2011)

MT 103 - VECTOR ALGEBRA AND CLASSICAL MECHANICS I (REPEAT)

Answer all Questions | Time: Three hours

1. Define the term *colinear* vector.

   (a) If the direction of a differential vector function $\underline{a}(t)$ is constant, then show that
   $$\underline{a} \wedge \frac{d\underline{a}}{dt} = \underline{0}.$$

   (b) Find an equation for the plane perpendicular to the vector $\underline{A} = 2\underline{i} + 3\underline{j} + 6\underline{k}$ and passing through the terminal point of the vector $\underline{B} = \underline{i} + 5\underline{j} + 3\underline{k}$. Hence find the distance from the origin to the plane.

   (c) If $\underline{u} = t^2\underline{i} - t\underline{j} + (2t+1)\underline{k}$ and $\underline{v} = (2t-3)\underline{i} + \underline{j} - t\underline{k}$, then find $\frac{d}{dt}(\underline{u} \cdot \underline{v})$ and $\frac{d}{dt}(\underline{u} \wedge \underline{v})$, when $t = 1$.

   (d) If $\frac{d\underline{a}}{dt} = \underline{c} \wedge \underline{a}$ and $\frac{d\underline{b}}{dt} = \underline{c} \wedge \underline{b}$, then prove that $\frac{d}{dt}(\underline{a} \wedge \underline{b}) = \underline{c} \wedge (\underline{a} \wedge \underline{b})$.

2. (a) Let $\underline{r}$ be a position vector and $|\underline{r}| = r$. Prove the following:

   i. $\hat{\underline{r}} \wedge d\hat{\underline{r}} = \frac{\underline{r} \wedge d\underline{r}}{r^2}$;

   ii. If $\underline{r} \cdot d\underline{r} = 0$, then $r =$ constant.

   (b) Define the term *gradient* of the scaler field ϕ.

   i. Show that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x, y, z) = c$, where c is a constant.

ii. Find the unit vector normal to the surface $x^2 - y^2 + z = 2$ at the point $(1, -1, 2)$.

iii. Find the directional derivatives of the function $f(x, y, z) = x^2 - y^2 + 2z^2$ at the point $P(1, 2, 3)$ in the direction of the line PQ , where Q is the point $(5, 0, 4)$.

3. Define the terms *divergence* and *curl* of the vector field $\underline{F}$.

(a) Let $\underline{F} = \underline{F}(x, y, z)$ be a vector function and $\phi = \phi(x, y, z)$ be a scalar function. Prove that

$$\operatorname{div}(\phi \underline{F}) = \phi \operatorname{div} \underline{F} + \operatorname{grad}\phi \cdot \underline{F}.$$

Hence show that

$$\operatorname{div}(r^n \underline{r}) = (n + 3)r^n.$$

(b) Define the term *solenoidal* vector.

Show that the vector

$$\underline{A} = (x + 3y)\underline{i} + (y - 3z)\underline{j} + (x - 2z)\underline{k}$$

is solenoidal and find $(\underline{A} \cdot \underline{\nabla})\underline{A}$.

(c) Define the term *irrotational* vector.

Determine the constants a, b and c so that the vector

$$\underline{F} = (x + 2y + az)\underline{i} + (bx - 3y - z)\underline{j} + (4x + cy + 2z)\underline{k}$$

is irrotational.

4. State the *Stoke's theorem* and *Green's theorem*.

(a) If $\underline{F} = (2x + y)\underline{i} + (3y - x)\underline{j}$, evaluate $\int_C \underline{F} \cdot d\underline{r}$ where C is the curve in the $xy-$plane consisting of the straight line from $(0, 0)$ to $(2, 0)$ and then to $(3, 2)$.

(b) Use the *Green's theorem* to evaluate $\int (x^2 - 2xy)dx + (x^2y + 3)dy$, around the boundary of the region defined by $y^2 = 8x$ and $x = 2$.

(c) Use the *Stoke's theorem* to evaluate $\int\int_S (\underline{\nabla} \wedge \underline{A}) \cdot \underline{n}\, ds$, where $\underline{A} = (x - z)\underline{i} + (x^3 + yz)\underline{j} - 3xy^2\underline{k}$ and S is the surface of the cone $z = 2 - \sqrt{x^2 + y^2}$ above the xy plane.

5. Obtain the radial and transverse components of the velocity and acceleration of a particle in the polar co-ordinate system.

A particle of mass m rests on a smooth horizontal table attached through a fixed point on the table by a light elastic string of modules mg and unstretched length 'a'. Initially a string is just taut and the particle is projected along the table in a direction perpendicular to the line of the string with velocity $\sqrt{\frac{4ag}{3}}$. Prove that if r is the distance of the particle from the fixed point at time t then

$$\frac{d^2r}{dt^2} = \frac{4ga^3}{3r^3} - \frac{g(r-a)}{a}$$

Prove that the string will extend until its length is $2a$ and that the velocity of the particle is then half of its initial velocity.

6. A particle moves in a plane with the velocity v and the tangent to the path of the particle makes an angle ψ with a fixed line in the plane. Write the velocity and acceleration components of the particle in intrinsic coordinate. Using these, show that the components of acceleration along the tangent and perpendicular to it are given by $v\frac{dv}{ds}$ and $v^2\frac{d\psi}{ds}$, respectively.

A smooth wire in the form of an arc of a cycloid $s = 4a \sin\psi$, is fixed in a vertical plane with the vertex downwards and the tangent at the vertex is horizontal. A small bead of mass m is threaded on the wire and is projected from the vertex with speed $\sqrt{8ag}$. If the resistance of the medium in which the motion takes place is $mv^2/8a$, where v is the speed, then show that the bead comes to instantaneous rest at a cusp ($\psi = \pi/2$) and returns to the starting point with speed $\sqrt{8ag(1 - 2e^{-1})}$.