EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SECOND EXAMINATION IN SCIENCE - (2009/2010)

FIRST SEMESTER (June/July, 2011)

MT 203 - EIGENSPACES AND QUADRATIC FORMS

Answer all questions **Time: Two hours**

1. (a) Define the following terms as applied to a square matrix $A = (a_{ij})$:

   i. eigenvalue;

   ii. characteristic polynomial, $\psi_A(\lambda)$, of A;

   iii. trace of A $(tr(A))$.

   (b) Let x be an eigenvector of a real $n \times n$ matrix A corresponding to the eigenvalue λ. Show that x is an eigenvector corresponding to the eigenvalue λ^m of A^m, for each $m = 1, 2, 3, \cdots$. Hence show that, if A is

   i. an idempotent matrix, then λ must be 0 or 1.

   ii. a nilpotent matrix, then $\psi_A(t) = t^n$ and $tr(A) = 0$.

   (c) Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be eigenvalues of an $n \times n$ matrix A with multiplicities. Prove the following:

   i. $\lambda_j = a_{jj} + \sum\limits_{i \neq j}(a_{ii} - \lambda_i)$, for $j = 1, 2, \cdots, n$;

   ii. $\det A = \lambda_1 \times \lambda_2 \times \cdots \times \lambda_n$, where $\det A$ means determinant of A.

   (d) Prove that, if two diagonalizable matrices A and B have the same eigenvectors then, $AB = BA$.

   Prove the converse of the above statement with an assumption that the eigenvalues of A are all distinct.

2. (a) Define the following terms:

   i. minimum polynomial;

   ii. irreducible polynomial,

   of a square matrix.

   (b) Prove the following:

   i. If $m(t)$ is the minimum polynomial of an $n \times n$ matrix A and $\psi_A(t)$ characteristic polynomial of A, then $\psi_A(t)$ divides $[m(t)]^n$.

   ii. The characteristic and minimum polynomials of a square matrix ha same irreducible factors.

   iii. $f(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$ is the minimum polynomial $n-$square matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & \cdots & 0 & 0 & -a_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 0 & 1 & -a_{n-1} \end{pmatrix}.$$

   Hence find the matrix whose minimum polynomial is $t^4 - 5\,t^3 - 2\,t^2 + 7$

3. (a) Find an orthogonal transformation which reduces the following quadratic to a diagonal form

$$5x_1^2 + 11x_2^2 - 2x_3^2 + 12x_1x_3 + 12x_2x_3.$$

   (b) Simultaneously diagonalize the following pair of quadratic forms

$$x_1^2 - x_2^2 + x_3^2 - 2x_2x_3 - 2x_1x_3 - 2x_1x_2;$$

$$3x_1^2 + x_2^2 + 3x_3^2 + 2x_1x_2 - 2x_2x_3 - 2x_1x_3.$$

4. (a) Prove that A is an $n \times n$ real symmetric matrix if and only if there exists an orthogonal matrix Q such that $Q^T AQ$ is diagonal.

Find an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$, where

$$A = \begin{pmatrix} -2 & 4 & -2 \\ 4 & 4 & -4 \\ -2 & -4 & 5 \end{pmatrix}.$$

(b) Define the term inner product in a vector space.

Let $C[0,1]$ be the vector space of all real-valued continuous functions on $[0,1]$.

For any two functions $f(x)$ and $g(x)$ in $C[0,1]$, define

$$\langle f, g \rangle = \int_0^1 f(x)g(x)dx.$$

Show that $\langle \ , \ \rangle$ is an inner product on $C[0,1]$.

(c) Use the Gram-Schmidt process to find orthonormal basis for the column space of the matrix

$$\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 1 & 0 & 4 \\ 1 & 1 & 0 \end{pmatrix}.$$