EASTERN UNIVERSITY, SRI LANKA

FIRST EXAMINATION IN SCIENCE - 2005/2006 & 2006/2007

(Aug./Sep.' 2007)

FIRST SEMESTER

ST 101 - PROBABILITY THEORY

(proper & Repeat)

Answer all questions

Time : Three hours

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1. (a) Define the term "probability function".

   (i) Show that the probability that exactly one of the events A or B occurs is
   $$P(A) + P(B) - P(A \cap B).$$

   (ii) Prove that
   $$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$$
   where $A_i (i = 1, 2, \cdots, n)$ are events defined on the sample space.

   (b) Explain briefly the "Conditional probability".

   (i) Let A and B be two events. Show that
   $$P(A/B^c) = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$
   Hence show that $P(A \cap B) \geq P(A) + P(B) - 1$.

   (ii) Let A and B be two events. Show that, with the usual notation,
   $P(A) = P(A/B)P(B) + P(A/B^c)P(B^c)$.

   (iii) Tomorrow there will be either rain or snow but not both, the probability of rain is $\frac{2}{5}$ and the probability of snow is $\frac{3}{5}$. If there is rain then the probability that I will be late for my lecture is $\frac{1}{5}$, while the corresponding probability in the event of snow is $\frac{3}{5}$. What is the probability that I will be late ?

2. (a) A random variable X has a negative binomial distribution with parameters k and θ. Its probability mass function is given by

$$P(X=x)=\begin{cases}\binom{x-1}{k-1}\theta^k(1-\theta)^{x-k} & ; x=k,k+1,k+2,\cdots\cdots \\ 0 & ; \text{o/w}\end{cases}$$

Find the moment generating function of a random variable X and Use the moment generating function derived to show that the negative binomial distribution has mean, $\mu=\frac{k}{\theta}$ and variance, $\sigma^2=\frac{k(1-\theta)}{\theta^2}$

(b) An item is produced in large numbers. The machine is known to produce 5% defectives. A quality control inspector is examining the items by taking them at random. What is the probability that at least 4 items are to be examined in order to get 2 defectives?

(c) If X and Y are independent Poisson variates with means λ_1 and λ_2 respectively, find the probability that

(i) $X+Y=k$

(ii) $X=Y$

3. (a) A business convention holds its registration on Wednesday morning from 9:00 a.m. until 12:00 noon. Past history has shown that registrant arrivals follow a Poisson distribution at an average rate of 1.8 every 15 seconds. Fortunately, several facilities are available to register convention members.

(i) What is the average number of seconds between arrivals to the registration area for this conference based on past results?

(ii) What is the probability that 25 seconds or more would pass between registration arrivals?

(iii) What is the probability that less than 5 seconds will elapse between arrivals?

(iv) Suppose the registration computers went down for a 1 - minute period. Would this condition pose a problem? What is the probability that at least 1 minute will elapse between arrivals?

(b) Suppose that in the bookkeeping operation of a large corporation the probability of a recording error on any one billing is 0.005. Suppose the probability of a recording error from one billing to the next is constant, and 1,000 billings are randomly sampled by an auditor.

(i) What is the probability that fewer than four billings contain a recording error?

(ii) What is the probability that more than 10 billings contain a billing error?

(iii) What is the probability that all 1,000 billings contain no recording errors?

4. (a) If X is a random variable having a Binomial distribution with the parameters n and θ then show that the moment generating function of $Z = \dfrac{X - n\theta}{\sqrt{n\theta(1-\theta)}}$ approaches that of the standard normal distribution when $n \longrightarrow \infty$.

(b) In an examination it is laid down that a student passes if he secures 30 percent or more marks. He is placed in the first, second or third division according as he secures 60% or more marks, between 45% and 60% marks and marks between 30% and 45% respectively. He gets distinction in case he secures 80% or more marks. It is noticed from the result that 10% of the students failed in the examination, whereas 5% of them obtained distinction. Calculate the percentage of students placed in the second division. (Assume normal distribution of marks.)

(c) The mean yield for one - acre plot is 662 kilos with a standard deviation 32 kilos. Assuming normal distribution, how many one - acre plots in a batch of 1000 plots would you expect to have yield

(i) over 700 kilos,

(ii) below 650 kilos,

(iii) what is the lowest yield of the best 100 plots?

5. (a) State and prove **"Baye's Theorem"**.

(b) The contents of urns I, II and III are as follows:

1 white, 2 black and 3 red balls,

2 white, 1 black and 1 red balls, and

4 white, 5 black and 3 red balls.

One urn is chosen at random and two balls drawn. They happen to be white and red. What is the probability that they come from urns I, II or III?

(c) Among the 300 employees of a company, 240 are union members, whereas the others are not. If 6 of the employees are chosen by lot to serve on a committee that administers the pension fund, find the probability that 4 of the 6 will be union members using

(i) The formula for the hypergeometric distribution;

(ii) The binomial distribution as an approximation.

6. (a) Define what is meant by **"Random variable"**.

Let X be a continuous random variable and let a and b be constants. Show that,

$E(aX + b) = aE(X) + b$ and $Var(aX + b) = a^2 Var(X)$.

(b) One of the earliest applications of the Poisson distribution was in analyzing incoming calls to a telephone switch-board. Analysts generally believe that random phone calls are Poisson distributed. Suppose phone calls to a switch-board arrive at an average rate of 2.4 calls per minute.

(i) If an operator wants to take a 1-minute break, what is the probability that there will be no calls during a 1-minute interval?

(ii) If an operator can handle at most five calls per minute, what is the probability that the operator will be unable to handle the calls in any 1-minute period?

(iii) What is the probability that exactly three calls will arrive in a 2-minute interval?

(iv) What is the probability that one or fewer calls will arrive in a 15-second interval?

(c) If t is a any positive real number, show that the function given by $P(X = x) = e^{-t}(1 - e^{-t})^{x-1}$ can represent a probability mass function of a random variable X assuming the values $1, 2, 3, \cdots$, find the $E(X)$ and $Var(X)$ of the distribution.