23 AUG 2013
EASTERN UNIVERSITY, SRI LANKA

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DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE -2010/2011

FIRST SEMESTER (Nov./Dec, 2012)

MT 103 - VECTOR ALGEBRA AND CLASSICAL MECHANICS I

(RE-REPEAT)

Answer all Questions   Time: Three hours

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1. Define the terms *colinear vectors* and *coplanar vectors*.

   (a) Prove the following:

      i. the diagonals of a parallelogram bisect each other;

      ii. If $x\underline{a} + y\underline{b} + z\underline{c} = \underline{0}$, then $x = y = z = 0$, where $\underline{a}, \underline{b}$ and $\underline{c}$ are non-coplanar vectors.

   (b) Let $\underline{a}, \underline{b}$ and $\underline{c}$ be three vectors such that $\underline{a}$ is perpendicular to both $\underline{b}$ and $\underline{c}$, and $|\underline{b}| = |\underline{c}|$. Show that the equation of the plane through the three points whose position vectors are $\underline{a}, \underline{b}$ and $\underline{c}$, is

$$\left\{\frac{\underline{a}}{|\underline{a}|^2} + \frac{\underline{b}+\underline{c}}{|\underline{b}||\underline{c}| + \underline{b}\cdot\underline{c}}\right\}\cdot\underline{r} = 1.$$

   Hence find the equation of the plane through the points $(2, -1, 1,), (3, 2, -1), (-1, 3, 2)$.

2. Define the term *divergence* of the vector field $\underline{F}$.

(a) Let $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ and $|\underline{r}| = r$. Show that

i. $\text{div}(r^n\underline{r}) = (n+3)r^n$, for $n \in \mathbb{Z}$;

ii. $\text{div}\left(\frac{\underline{r}}{r^3}\right) = 0$.

(b) Define the term *gradient* of the scaler field ϕ.

i. Show that $\underline{\nabla}\phi$ is a vector perpendicular to the surface $\phi(x, y, z) = c$, where c is a constant.

ii. Find the directional derivatives of the function, $f(x, y, z) = x^2 - y^2 + 2z^2$, at the point $P(1, 2, 3)$ in the direction of the line PQ , where Q is the point $(5, 0, 4)$.

(c) Prove the following:

i. if $\underline{a}$ and $\underline{b}$ are irrotational vectors, then $\underline{a} \wedge \underline{b}$ is a solenoidal vector,

ii. if $\underline{r} \wedge d\underline{r} = \underline{0}$, then $\hat{\underline{r}} =$ constant.

3. State the *Stoke's theorem* and *Green's theorem.*

(a) If $\underline{F} = (2x + y)\underline{i} + (3y - x)\underline{j}$, then evaluate $\int_C \underline{F} \cdot d\underline{r}$, where C is the curve in the xy-plane consisting of the straight line from $(0, 0)$ to $(2, 0)$ and then to $(3, 2)$.

(b) If S is any open surface bounded by a simple closed curve C and $\underline{B}$ is any vector, then prove that

$$\oint_C d\underline{r} \wedge \underline{B} = \int\int_S (\underline{n} \wedge \underline{\nabla}) \wedge \underline{B}\ ds.$$

(c) Verify the *Stoke's theorem* for $\underline{A} = (x - z)\underline{i} + (x^3 + yz)\underline{j} - 3xy^2\underline{k}$ and S is the surface of the cone $z = 2 - \sqrt{x^2 + y^2}$ above the xy-plane.

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4. Obtain the radial and transverse components of the velocity and acceleration of a particle in the polar co-ordinate system.

Masses m_1, m_2 are attached to the ends of a weightless inextensible string AOB and rest on a smooth fixed peg at O, and the portions $OA(= x)$, and $OB(= y)$ of the string are in a straight line. The mass m_1 is now projected horizontally with velocity v perpendicular to OA. If the string remains in contact with the peg, and all the motion takes place in a horizontal plane, prove that the mass m_2 reaches the peg with velocity

$$\frac{v}{(x+y)}\sqrt{\frac{m_1 y(2x+y)}{(m_1+m_2)}}.$$

5. A particle moves in a plane with the velocity v and the tangent to the path of the particle makes an angle ψ with a fixed line in the plane. Write the velocity and acceleration components of the particle in intrinsic coordinate. Using these, show that the components of acceleration along the tangent and perpendicular to it are given by $v\dfrac{dv}{ds}$ and $v^2\dfrac{d\psi}{ds}$, respectively.

A particle is projected horizontally with the velocity v_0 in a medium which offers a resistance kv^2 per unit mass, where v is the speed. Let ψ be the downward inclination of its path to the horizontal after it has traveled a distance s on the arc. Show that

(a) $v\cos\psi = v_0 e^{-ks}$;

(b) $e^{2ks}\dfrac{ds}{d\psi} = \dfrac{v_0^2}{g}\sec^3\psi$;

by resolving the velocity components to the horizontal and perpendicular direction to the path.

Hence find the intrinsic equation of the particle path.

6. With usual notations, obtain the equation of motion of a rocket of varying mass in the form

$$\underline{F}(t) = m(t)\frac{d\underline{v}}{dt} + \underline{v_0}\frac{dm(t)}{dt}.$$

A rocket is fired upwards and the matter is ejected with constant relative velocity gT at a constant rate $\frac{2M}{T}$. Initially the mass of the rocket is $2M$, half of this is available for ejection. Neglecting air resistance and variation in gravitational attraction, show that the greatest speed of the rocket is attained when the mass of the rocket is reduced to M, and determine this speed.

Show also that the rocket will reach the greatest height given by

$$\frac{1}{2}\,gT^2(1-\ln 2)^2.$$