EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SECOND EXAMINATION IN SCIENCE - 2010/2011

FIRST SEMESTER (March/April, 2013)

AM 207 - NUMERICAL ANALYSIS

(Proper & Repeat)

Answer all questions **Time : Two hours**

1. (a) Define the terms:

   i. absolute error;

   ii. relative error of a numerical value.

   (b) Let $f(x) = \sin 2x + \cos 2x$.

   i. Write down the second Taylor polynomial $P_2(x)$ of $f(x)$ centered around $x_0 = 0$.

   ii. Write down the corresponding Taylor remainder $R_2(x)$.

   iii. Suppose $P_2(x)$ is used to approximate $f(x)$ on the interval $[-\pi, \pi]$. How large in magnitude can the absolute error in this approximation be?

2. (a) Let $x = \phi(x)$ be the rearrangement of the equation $f(x) = 0$ and define the iteration,

$$x_{n+1} = \phi(x_n), \qquad n = 0, 1, \ldots,$$

with the initial value x_0. If $\phi'(x)$ exists and is continuous such that $|\phi'(x)| \leq K < 1$ for all x, then show that the sequence (x_n) generated by the above iteration converges to the unique root α of the equation $f(x) = 0$.

Find a real root of the equation

$$f(x) = x^3 + x^2 - 1 = 0$$

by the method of iteration.

(b) i. Define the order and the asymptotic error constant of the iteration method to compute the non linear equation $f(x) = 0$.

ii. Show that the order of convergence of secant method is approximately 1.62.

Apply secant method to find a solution to the equation $x - \cos x = 0$ near $x = 0$ in the interval $\left[0, \frac{\pi}{2}\right]$ that is accurate within 10^{-4}.

3. (a) Let $f(x)$ be a $(n+1)$-times continuously differentiable function of x and y_0, $y_1, \ldots, y_n$ be the values of $f(x)$ at $x = x_0, x_1, \ldots, x_n$, respectively. Then derive the Lagrange's Interpolation polynomial $P_n(x)$ to estimate the value of $f(x)$ for any $x \in [x_0, x_n]$, in the form

$$P_n(x) = \sum_{i=0}^{n} \frac{\Pi(x) y_i}{(x - x_i)\Pi'(x_i)},$$

where $\Pi(x) = \prod_{i=0}^{n} (x - x_i)$.

(b) Consider the exponential function $y(x) = e^x$. Use the Lagrange interpolating polynomial to estimate $e^{0.2}$, using $y(x)$ at the nodes $(0, 0.1, 0.3)$. Estimate the maximum error in your approximation to $e^{0.2}$.

4. (a) Let $I(f) = \int_a^b f(x)dx$ and $I(P_n)$ is the approximation to $I(f)$, where $P_n(x)$ is the interpolating polynomial which interpolates $f(x)$ at equally spaced nodes $a = x_0, x_1, ..., x_n = b$, where $x_k = x_0 + kh$, for $k = 0, 1, ..., n$, and $x_i \in [a, b]$ for $i = 0, 1, ..., n$. Then the error in the approximation is given by

$$E(f) = I(f) - I(P_{n)}.$$

Obtain the composite Trapezoidal rule and show that the composite error is

$$-\frac{(b-a)}{12}h^2 y''(\xi), \text{ where } \xi \in [a, b].$$

Determine

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+x} dx$$

to an accuracy of $\epsilon = 10^{-2}$. using the Composite Trapezium Rule.

(b) Solve the following linear system of equations using Gaussian Elimination with partial pivoting:

$$\begin{aligned}
4x_1 + 4x_2 + x_3 + 4x_4 &= 12 ; \\
2x_1 + 5x_2 + 7x_3 + 4x_4 &= 1 ; \\
10x_1 + 5x_2 - 5x_3 \quad\quad &= 25 ; \\
-2x_1 - 2x_2 + x_3 - 3x_4 &= -10 .
\end{aligned}$$