EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
FIRST YEAR EXAMINATION IN SCIENCE - 2009/2010
SECOND SEMESTER - (APRIL/MAY, 2012)
MT 104 - DIFFERENTIAL EQUATIONS
AND
FOURIER SERIES
(REPEAT)

Answer All Questions Time Allowed: 3 Hours

Q1. (a) State the necessary and sufficient condition for the differential equation (DE)

$$M(x, y)\, dx + N(x, y)\, dy = 0$$

to be *exact*.

[10 Marks]

Prove that $x^a y^b$ is an integrating factor of the DE

$$kydx + lxdy + x^c y^d (mydx + nxdy) = 0$$

if

$$\frac{a+1}{k} = \frac{b+1}{l} \quad \text{and} \quad \frac{a+c+1}{m} = \frac{b+d+1}{n},$$

where $a, b, c, d, k(\neq 0), l(\neq 0), m(\neq 0)$ and $n(\neq 0)$ are constants. Hence integrate the DE

$$3ydx - 2xdy + x^2 y^{-1}(10ydx - 6xdy) = 0.$$

[50 Marks]

(b) Show that there are two values of the constant α for which α/x is a solution of the Riccati's equation

$$\frac{dy}{dx} - \frac{2}{x^2} + y^2 = 0, \qquad (1)$$

and hence obtain the general solution of the equation (1).

[40 Marks]

Q2. (a) If $F(D) = \sum_{i=0}^{n} p_i D^i$, where $D \equiv d/dx$ is a differential operator and p_i, $i = 1, \ldots, n$, are constants with $p_0 \neq 0$, then prove the following formulas:

(i) $\frac{1}{F(D)} e^{\alpha x} = \frac{1}{F(\alpha)} e^{\alpha x}$, α is a constant and $F(\alpha) \neq 0$;

(ii) $\frac{1}{F(D)} e^{\alpha x} V = e^{\alpha x} \frac{1}{F(D+\alpha)} V$, where V is a function only of x.

[30 Marks]

(b) Find the general solution of the following DEs by using the results in (a):

(i) $(D-1)y = (x+3)\,e^{2x}$;

(ii) $(D^2 - 5D + 6)y = x^3\,e^{2x}$.

[30 Marks]

(c) Show that

$$\frac{\cos \beta x - \cos(\beta + h)x}{(\beta + h)^2 - \beta^2}$$

is a solution of the DE

$$(D^2 + \beta^2)y = \cos(\beta + h)x,$$

where β and h are constants.
Hence deduce the particular integral of the DE

$$(D^2 + \beta^2)y = \cos \beta x.$$

[40 Marks]

Q3. (a) Prove, by induction or otherwise, that if $T \equiv x(d/dx) = xD$, where $D \equiv d/dx$, then

$$x^n \frac{d^n y}{dx^n} = T(T-1)(T-2)\ldots(T-n+1)y,$$

where n is a positive integer.
Hence or otherwise show that the solution of the DE

$$x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^5$$

is

$$y(x) = Ax^2 + Bx^3 + \frac{1}{6}x^5,$$

where A and B are arbitrary constants.

[70 Marks]

(You may use the following identities without proof:

(i) $F(T)x^m = x^m F(m)$;

(ii) $\dfrac{1}{F(T)}x^m = \dfrac{x^m}{F(m)}$, provided $F(m) \neq 0$, where m is some constant.)

(b) With $D \equiv d/dx$, solve the following system of DEs

$$\begin{aligned}(5D+4)y-(2D+1)z &= e^{-x},\\ (D+8)y-3z &= 5\,e^{-x}.\end{aligned}$$

[30 Marks]

Q4. (a) Define what is meant by the point, '$x = x_0$', being

(i) an *ordinary* ;

(ii) a *singular*;

(iii) a *regular singular*

point of the DE

$$y'' + p(x)y' + q(x)y = 0,$$

where the prime denotes differentiation with respect to x, and $p(x)$ and $q(x)$ are rational functions.

[30 Marks]

(b) (i) Find the regular singular point(s) of the DE

$$9x^2y'' + 9x^2y' + 2y = 0. \qquad (2)$$

(ii) Use the method of Frobenius to find the general solution of the equation (2).

[70 Marks]

Q5. (a) Solve the following system of DEs:

(i) $\dfrac{dx}{1} = \dfrac{dy}{3} = \dfrac{dz}{5z + \tan(y-3x)}$;

(ii) $\dfrac{dx}{y+z} = \dfrac{dy}{z+x} = \dfrac{dz}{x+y}$.

[30 Marks]

(b) Write down the condition of integrability of the total differential equation

$$P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = 0.$$

[5 Marks]

Hence solve the following equation

$$(y^2 + z^2 - x^2)\,dx - 2xy\,dy - 2xz\,dz = 0. \qquad (3)$$

(You may use the integrating factor $\mu = 1/x^2$ for equation (3).)

[15 Marks]

(c) Find the general solution of the following linear first-order partial differential equations:

(i) $xp + yq = z$;

(ii) $yp - xq = y^2 - x^2$.

[30 Marks]

(d) Apply Charpit's method or otherwise to find the complete and the singular solution of the following non-linear first-order partial differential equation

$$p^2 + yq + 2z + 2y^2 = 0.$$

Here, $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$.

[20 Marks]

Q6. (a) Show that the Fourier series of the function

$$f(x) = x^2, \quad -\pi \leq x \leq \pi$$

can be expressed as

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos n\pi}{n^2}.$$

[20 Marks]

Hence deduce that

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

[20 Marks]

(b) Use Fourier transform to solve the one-dimensional heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x > 0, \quad t > 0$$

subject to the conditions

$$u(0,t) = 0, \qquad u(x,0) = \begin{cases} 1, & 0 < x < 1, \\ 0, & x > 1 \end{cases}$$

and $u(x,t)$ is bounded.

[40 Marks]

(c) (i) Define the *gamma-function* $\Gamma(x)$ and *beta-function* $B(m,n)$, where m, n are positive integers.

(ii) Evaluate the integral

$$\int_0^1 \frac{1}{\sqrt{1-x^4}}\,dx.$$

(You may use the following results without proof

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \text{and} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.)$$

[20 Marks]

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