



EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE - 2015/2016

FIRST SEMESTER (August/September, 2018)

MT 1212 - ALGEBRA

Answer all questions

Time : Two hours

1. (a) For any integers a, b, c , prove the following:
- if $a|b$ and $a|c$, then $a^2|bc$;
 - $a|b$ if and only if $ac|bc$, where $c \neq 0$.
- (b) Prove the following properties of the greatest common divisor:
- if $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$, then $\gcd(a, bc) = 1$;
 - if $\gcd(a, b) = 1$ and $c|(a + b)$, then $\gcd(a, c) = \gcd(b, c) = 1$.
- (c) Determine all solutions in the positive integers of the following Diophantine equation

$$54x + 21y = 906.$$

2. (a) Solve the following linear congruences:
- $6x \equiv 15 \pmod{21}$;
 - $34x \equiv 60 \pmod{98}$.
- (b) Solve the following set of simultaneous congruences:

$$x \equiv -2 \pmod{5}$$

$$x \equiv 3 \pmod{2}$$

$$x \equiv 2 \pmod{7}$$

$$x \equiv -18 \pmod{11}.$$

3. (a) Define the term *group*.
(b) Show that the set \mathbb{Z} of all integers is an abelian group with respect to the operation $*$ defined by

$$a * b = a + b + 1 \quad \forall a, b \in \mathbb{Z}.$$

- (c) i. Show that the identity element of a group is unique.
ii. Show that a group G is an abelian group if $(ab)^2 = a^2b^2$ for all $a, b \in G$.
4. (a) Let H be a non-empty subset of a group G . If $ab^{-1} \in H$ for each $a, b \in H$, prove that H is a subgroup of G , where a^{-1} is the inverse of a .
(b) Let H be a subgroup of G and let $g \in G$. Show that

$$gHg^{-1} = \{ghg^{-1} : h \in H\}$$

is a subgroup of G .

- (c) Prove that if H and K are subgroups of a group G , then $H \cap K$ is a subgroup