



EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
THIRD EXAMINATION IN SCIENCE - 2010/2011
FIRST SEMESTER (April, 2013)
MT 302 - COMPLEX ANALYSIS
(PROPER/REPEAT)

Answer all Questions

Time: Three hours

- Q1. (a) Let $A \subseteq \mathbb{C}$ be an open set and let $f : A \rightarrow \mathbb{C}$. Define what is meant by f being **analytic** at $z_0 \in A$.
- (b) Let the function $f(z) = u(x, y) + iv(x, y)$ be defined throughout some ϵ neighborhood of a point $z_0 = x_0 + iy_0$. Suppose that the first - order partial derivatives of the functions u and v with respect to x and y exist everywhere in that neighborhood and that they are continuous at (x_0, y_0) . Prove that, if those partial derivatives satisfy the **Cauchy-Riemann** equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at (x_0, y_0) , then the derivative $f'(z_0)$ exists.

- (c) (i) Prove that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic.
- (ii) Find the function $v(x, y)$ such that $f(z) = u(x, y) + iv(x, y)$ is analytic.
- Q2. (a) (i) Define what is meant by a curve $\gamma : [\alpha, \beta] \rightarrow \mathbb{C}$ smooth.
- (ii) For a path γ and a continuous function $f : \gamma \rightarrow \mathbb{C}$, define $\int_{\gamma} f(z) dz$.
- (b) Prove that if γ is a path and $f \in C(\gamma)$, then $|f(z)| \leq M$, for all $z \in \gamma$ and $M \geq 0$ such that $\left| \int_{\gamma} f(z) dz \right| \leq ML$, where $L = \text{length}(\gamma)$.

(c) State the **Cauchy's Integral Formula**.

By using the **Cauchy's Integral Formula** compute the following integrals:

(i) $\int_{C(0;2)} \frac{z}{(9-z^2)(z+i)} dz;$

(ii) $\int_{C(0;1)} \frac{1}{(z-a)^k(z-b)} dz$, where $k \in \mathbb{Z}$, $|a| > 1$ and $|b| < 1$.

Q3. (a) State and prove the **Mean Value Property for Analytic Functions**.

(b) (i) Define what is meant by the function $f : \mathbb{C} \rightarrow \mathbb{C}$ being **entire**.

(ii) Let f be analytic on $D(z_0; r)$ and $0 < s < r$. Let

$$M(s) = \sup \{|f(z)| : |z - z_0| = s\}.$$

Prove that

$$|f^{(n)}(z_0)| \leq \frac{n!M(s)}{s^n},$$

and if $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$, then

$$|a_n| \leq \frac{M(s)}{s^n}.$$

(c) Prove the **Maximum - Modulus Theorem**: Let f be analytic in an open connected set A . Let γ be a simple closed path that is, contained together with its inside in A . Let

$$M := \sup_{z \in \gamma} |f(z)|.$$

If there exists z_0 inside γ such that $|f(z_0)| = M$, then f is constant throughout A . Consequently, if f is not constant in A , then

$$|f(z)| < M, \forall z \text{ inside } \gamma.$$

(State any theorem you use without proof)

Q4. (a) Let $\delta > 0$ and let $f : D^*(z_0; \delta) \rightarrow \mathbb{C}$, where

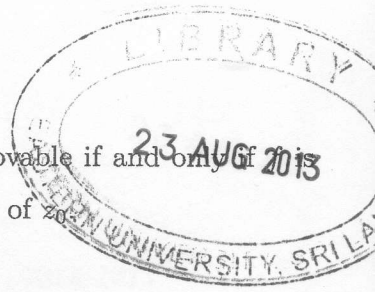
$$D^*(z_0; \delta) := \{z : 0 < |z - z_0| < \delta\}.$$
 Define what is meant by

(i) f having a singularity at z_0 ;

(ii) the order of f at z_0 ;

(iii) f having a pole or zero at z_0 of order m ;

(iv) f having a simple pole or simple zero at z_0 .



- (b) Prove that an isolated singularity z_0 of f is removable if and only if f is bounded on some deleted neighborhood $D^*(z_0; \delta)$ of z_0 .
- (c) Prove that if f has a simple pole at z_0 , then

$$\text{Res}(f; z_0) = \lim_{z \rightarrow z_0} (z - z_0)f(z).$$

Q5. Let f be analytic in the upper - half plane $\{z : \text{Im}(z) \geq 0\}$, except at finitely many points, none on the real axis. Suppose there exist $M, R > 0$ and $\alpha > 1$ such that

$$|f(z)| \leq \frac{M}{|z|^\alpha}, |z| \geq R \text{ with } \text{Im}(z) \geq 0.$$

Then prove that

$$I := \int_{-\infty}^{\infty} f(x) dx$$

converges (exists) and

$$I = 2\pi i \times \text{Sum of Residues of } f \text{ in the upper half plane.}$$

Hence evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{1 + x^2} dx.$$

(You may assume without proof the Residue Theorem).

- Q6. (a) State the **Argument Theorem**.
- (b) Prove **Rouche's Theorem** : Let γ be a simple closed path in an open starset A . Suppose that
- (i) f, g are analytic in A except for finitely many poles, none lying on γ .
 - (ii) f and $f + g$ have finitely many zeros in A .
 - (iii) $|g(z)| < |f(z)|, z \in \gamma$. Then

$$ZP(f + g; \gamma) = ZP(f; \gamma)$$

where $ZP(f + g; \gamma)$ and $ZP(f; \gamma)$ denote the number of zeros - number of poles inside γ of $f + g$ and f respectively, where each is counted as many times as its order.

- (c) State the **Fundamental theorem of Algebra**.
- (d) Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.