EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

EXTERNAL DEGREE

THIRD EXAMINATION IN SCIENCE-2008/2009

SECOND SEMESTER (Feb./Apr., 2015)

EXTMT 301 - GROUP THEORY

Answer all questions **Time : Three hours**

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1. State what is meant by

   - *a group G*;
   - *a subgroup H of a group G.*

   (a) Let H be a non empty subset of a group G. Prove that H is a subgroup of G if only if $ab^{-1} \in H$ for all $a, b \in H$.

   (b) Let H be a non-empty subset of a group G. Prove that H is a subgroup of G if and only if $HH^{-1} = H$.

   (c) Let H and K be two subgroups of a group G. Prove that HK is a subgroup of G if and only if $HK = KH$.

2. State and prove the *Lagrange's theorem* for a finite group.

   (a) In a group G, H and K are different subgroups of order p, where p is a prime number. Prove that $H \cap K = \{e\}$, where e is the identity element of G.

(b) Let G be a non-abelian group of order 10. Prove that G contains at least one element of order 5.

(c) If every non-identity element of a group G has order 2, then show that G is abelian.

3. State what is meant by a *normal subgroup* of a group G.

(a) Let $\phi : G \longrightarrow G_1$ be a homomorphism of a group G onto a group G_1. Prove the following:

i. $\ker\phi = \{g \in G \mid \phi(g) = e_1\}$ is a normal subgroup of G, where e_1 is an identity element of G_1;

ii. if H is a normal subgroup of G, then $\phi(H)$ is a normal subgroup of G_1.

(b) Let G be a group. Prove that for any non-empty subset H of G,
$N(H) = \{x \in G \mid xH = Hx\}$ is a subgroup of G.
For any subgroup H of G, prove the following:

i. H is a normal subgroup of $N(H)$;

ii. $N(H)$ is the largest subgroup of G in which H is normal;

iii. H is a normal subgroup of G if and only if $N(H) = G$.

4. (a) State and prove the *first isomorphism theorem*.

(b) Let H and K be two normal subgroups of a group G such that $K \subseteq H$. Prove that

i. $K \trianglelefteq H$;

ii. $\dfrac{H}{K} \trianglelefteq \dfrac{G}{K}$;

iii. $\dfrac{G/K}{H/K} \cong \dfrac{G}{H}$.

5. (a) Define what is meant by the *p−group*.

   Prove the following:

   i. every subgroup of a $p-$group is a $p-$group;

   ii. the homomorphic image of a $p-$group is a $p-$group.

   (b) Let G' be the commutator subgroup of a group G. Prove the following:

   i. G is abelian if and only if $G' = \{e\}$, where e is the identity element of G;

   ii. G' is a normal subgroup of G;

   iii. G/G' is abelian;

   iv. if H is a normal subgroup of G then G/H is abelian if and only if $G' \subseteq H$.

6. (a) Define the following terms as applied to a permutation group:

   i. *cyclic of order r*;

   ii. *transposition*;

   iii. *signature*.

   (b) Prove that the permutation group on n symbols S_n is a finite group of order $n!$.

   Is S_n abelian for $n > 2$? Justify your answer.

   (c) Prove that every permutation in S_n can be expressed as a product of disjoint cycles.

   (d) Express the permutation,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 5 & 7 & 4 & 2 & 8 & 1 & 6 \end{pmatrix}$$

   as a product of disjoint cycles. Hence or otherwise determine whether σ is even or odd.