EASTERN UNIVERSITY, SRI LANKA

FIRST EXAMINATION IN SCIENCE - 2016/2017

SECOND SEMESTER (March, 2019)

PM 102 - REAL ANALYSIS

Answer all Questions | Time: Three hours

Q1. (a) i. Define the terms '*Supremum*' and '*Infimum*' of a non-empty subset of $\mathbb{R}$.

ii. State the completeness property of $\mathbb{R}$, and use it to prove that every non-empty bounded below subset of $\mathbb{R}$ has an Infimum.

(b) Prove that an upper bound u of a non-empty bounded above subset S of $\mathbb{R}$ is the Supremum of S if and only if for every $\epsilon > 0$, there exist an $x \in S$ such that $x > u - \epsilon$.

(c) Find the Supremum and Infimum of the set

$$S = \left\{ \frac{2}{3}\left(1 - \frac{1}{10^n}\right) \;:\; n \in \mathbb{N} \right\}.$$

Q2. (a) State what is meant by a sequence of real numbers (x_n) converges to a limit a.

(b) Prove that every convergent sequence of real numbers is bounded.

(c) State the *Monotone Convergent Theorem.*

Let $x_1 = 1$ and $x_{n+1} = \frac{1}{4}(2x_n + 3)$ for all $n \in \mathbb{N}$.

i. Show that (x_n) is strictly increasing sequence.

ii. Show that $x_n \leq 2$ for all $n \in \mathbb{N}$.

iii. Does the sequence converge at all? Justify your answer.

Q3. (a) Define the following terms:

i. a *subsequence of a sequence*;

ii. *Cauchy sequence*.

(b) State and prove the *Bozano-Weierstrass Theorem.*

(c) Prove that a sequence (x_n) of real numbers is Cauchy if and only if it is co
vergent .

(d) Let (a_n) and (b_n) be two Cauchy sequences and $c_n = |a_n - b_n|$. Show that $(c$
is a Cauchy sequence.

Q4. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Explain what is meant by the function f has
limit $l(\in \mathbb{R})$ at a point $a(\in \mathbb{R})$.

(b) If $\lim_{x \longrightarrow a} f(x) = l$, then show that $\lim_{x \longrightarrow a} |f(x)| = |l|$. Is the converse true? Justif
your answer.

(c) Let f, g, and h be three real-valued functions defined on $A \subseteq \mathbb{R}$. Assume tha
for all $x \in A$, we have

$$f(x) \leq g(x) \leq h(x)$$

and that for $x_0 \in A$ we have $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = l$. Prove that

$$\lim_{x \to x_0} g(x) = l.$$

Hence show that $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x}\right) = 0.$

Q5. (a) Define what it means to say that a real-valued function f is continuous at a
point $'a'$ in its domain.

Let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0; \\ 1 & \text{if } x = 0. \end{cases}$$

Prove that, f is continuous at $x = 0$.

(b) Prove that if a function $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$, then it is bounded on $[a, b]$.

(c) State the *Intermediate Value Theorem* and use it to show that the equation $2x^2(x+2)-1 = 0$ has a root in each of the intervals $(-2, -1)$, $(-1, 0)$ and $(0, 1)$.

Q6. (a) i. Define what it means to say that the real-valued function f is differentiable at a point $'a'$ in its domain.

ii. Prove that every differentiable function is continuous. Is the converse true? Justify your answer.

(b) State the *Mean-Value Theorem* and use it to prove

$$x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}}, \quad \forall x \in (0, 1).$$

(c) Suppose that f and g are continuous on $[a, b]$ differentiable on (a, b) and $g'(x) \neq 0$ for all $x \in (a, b)$. Prove that there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$