EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE - 2010/2011

SECOND SEMESTER (June, 2013)

PM 107 - THEORY OF SERIES

(PROPER & REPEAT)

Answer all Questions Time: Two hours

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1. (a) State the necessary and sufficient condition for the convergence of a series of positive real numbers $\sum_{n=1}^{\infty} u_n$. [10 marks]

   (b) If $\lambda > 1$, prove that the series $\sum_{n=1}^{\infty} \frac{1}{n^{\lambda}}$ is convergent. [25 marks]

   (c) State the integral test for the series of non-negative terms.

   Hence show that the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ is convergent if $p > 1$ and divergent if $0 < p \leq 1$. [35 marks]

   (d) Test the convergence of the following series:

   i. $1 + \frac{x}{1!} + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \cdots$, for $x > 0$;

   ii. $\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^2} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots$. [30 marks]

   (You may state the convergence tests without prove)

2. (a) Define the following

   i. absolutely convergent series;

   ii. conditionally convergent series. [10 marks]

   (b) If $\sum_{n=1}^{\infty} u_n$ is an absolutely convergent series then prove that the series of its positive terms and the series of its negative terms are both convergent.

   [20 marks]

(c) Prove that every re-arrangement of an absolutely convergent series is convergent and the sum also does not change.

Is it true that the re-arrangement of an conditionally convergent series has the same sum? Justify your answer. [35 marks]

(d) State the **Leibnitz's** theorem for an infinite series of real numbers.

Hence show that the series

$$x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots,$$

converges only if $-1 \leq x \leq 1$. [35 marks]

3. (a) State the **Cauchy's** general principle of convergence for series. [10 marks]

Hence show that the series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ does not converge. [15 marks]

(b) State **Abel's** theorem for an infinite series of real numbers. [10 marks]

Use the above theorem to test the convergence of the series

$$\sum_{n=2}^{\infty} \frac{(n^3+1)^{\frac{1}{3}} - n}{\ln n}.$$

[15 marks]

(c) i. Show that

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \quad \text{for } |x| < 1.$$

[30 marks]

ii. Use the result in part(i), and the **Abel's** theorem for the power series to show that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

[20 marks]

4. (a) Define what is meant by the convergent of an infinite series of complex numbers $\sum_{n=1}^{\infty} z_n$. [10 marks]

(b) Show that the geometric series, $1 + z + z^2 + z^3 + \cdots$, has the sum $\frac{1}{1-z}$, when $|z| < 1$.

Hence find the sum of the series $\sum_{n=0}^{\infty} (n+1)z^n$. [25 marks]

(c) If $\sum_{n=1}^{\infty} z_n$ is an infinite series of complex numbers such that $\lim_{n\to\infty} \sqrt[n]{|z_n|} = l$, then prove that, if $l < 1$, the series converges absolutely and if $l > 1$, the series diverges.

Hence check whether the series $\sum_{n=0}^{\infty} \left(\frac{1}{2+i}\right)^n$ converges or diverges.

[35 marks]

(d) Show that the infinite series

$$1 + \frac{z}{1+z} + \frac{z^2}{(1+z)^2} + \cdots$$

converges for $z \in E$, where $E = \left\{ z \in \mathbb{C} : \operatorname{Re}(z) > -\frac{1}{2} \right\}$.

Hence find its sum. [30 marks]