EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SPECIAL DEGREE EXAMINATION IN MATHEMATICS

ACADEMIC YEAR - 2010/2011 (JUNE, 2016)

MTS 10 - NUMERICAL LINEAR ALGEBRA

Answer all Questions | Time: Three hours

1. (a) Define the term "*elementary lower-triangular matrix*".

   Let A be an $n \times n$ matrix and A_r be the principle sub-matrix of A of order $r(r \leq n)$. Prove that if $\det A_r \neq 0$ for $r < n$, then there exists a decomposition $A = LU$, where L is a product of elementary lower triangular matrices and U is an upper triangular matrix.

   (b) Define the term "*positive definite* "as applied to an $n \times n$ Hermitian matrix A. Prove that an Hermitian positive definite matrix A can be uniquely expressed as $A = LU$ where L is a unit lower-triangular matrix and U is an upper-triangular matrix.

   (c) Show that an Hermitian matrix A is positive definite if and only if $A = GG^H$ where G is a non-singular lower-triangular matrix. Determine G such that

$$GG^H = \begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 5 & 2 & -3 \\ 0 & 2 & 5 & 1 \\ 1 & -3 & 1 & 4 \end{bmatrix}.$$

Q2. (a) Define the term "*elementary Hermitian matrix*".

Prove that any product of $n \times n$ elementary Hermitian matrices is

matrix.

(b) Show that , for any real vector x ,there is a real elementary Hermiti

$H(\omega)$ such that $H(\omega)\, x \;=\; c\, e_1$, where $c^2 = x^T x$ and $e_1 \;=\; (1,0$

What is the optimal choice of the sign of c for the computation of ω

(c) Determine an upper triangular matrix U such that $HA = U$, whe

product of elementary Hermitian matrices and

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 4 & -1 \\ 2 & 5 & 0 \end{bmatrix},$$

making the optimal choice of sign in each stage of the process.

$Ax \;=\; b$ where $b \;=\; (5, 0, -1)^T$.

Q3. (a) Define the term "*spectral radius*" of an $n \times n$ matrix .

Let $\rho(A)$ denote the spectral radius of an $n \times n$ matrix A. Show t

$\epsilon > 0$, there is a matrix norm such that

$$\rho(A) \leq \| A \| < \rho(A) + \epsilon.$$

Hence show that if $\rho(A) < 1$ then $I - A$ is non singular and $\|$

$\dfrac{1}{1 - \| A \|}$ for some matrix norm.

(b) Let A be a non-singular matrix and E a matrix such that $\| A^{-}$

for some matrix norm subordinate to a vector norm. Let $Ax =$

Suppose that $(A + E)z \;=\; r + e$, where $r = b - Ay$ and y a

vectors. Show that $x - (y + z) = (A + E)^{-1}[E(x - y) - e]$ and

$$\frac{\| x - (y + z) \|}{\| x \|} \;\leq\; \frac{K(A)}{1 - K(A)\,\frac{\| E \|}{\| A \|}} \left[\frac{\| E \|}{\| A \|} \cdot \frac{\| x - y \|}{\| x \|} \right.$$

where $K(A)$ is the condition number of A.

(a) Define the term "*strictly diagonally dominant*" as applied to an $n \times n$ matrix A.

Prove that a strictly diagonally dominant matrix is non singular.

(b) Let $A = I - L - U$ be a strictly diagonally dominant, where L is strictly lower triangular and U is strictly upper triangular matrices. For arbitrary initial guess $x^{(0)}$, a sequence $\{x^{(r)}\}$ is defined by

$$x^{(r+1)} = (I - \omega L)^{-1}\left[\omega b + ((1-\omega)I + \omega U)x^{(r)}\right], \quad r = 0, 1, 2\dots .$$

Show that

$$x - x^{(r+1)} = M(x - x^{(r)}), \quad r = 0, 1, 2...,$$

where $M = (I - \omega L)^{-1}[(1-\omega)I + \omega U]$ and x is the solution of $Ax = b$. If $0 < \omega \leq 1$, show that the sequence $x^{(r)}$ converges to x.

The following equations are to be solved by Gauss-Seidel iteration (Successive Over -Relaxation with a parameter $\omega = 1$):

$$\begin{array}{rcrcrcrcl} 4x_1 & & & + & x_3 & + & x_4 & = & 1 \\ x_1 & & & + & 4x_3 & & & = & 3 \\ x_1 & + & x_2 & & & + & 4x_4 & = & 4 \\ & & 4x_2 & & & + & x_4 & = & 2 \end{array}$$

Starting with $x^{(0)} = 0$ and using four significant digit arithmetic , obtain $x^{(1)}$ and $x^{(2)}$.

(a) Define the term "upper Hessenberg matrix".

Let A be an $n \times n$ matrix . Show that there exists a unitary matrix S, a product of elementary Hermitian matrices, such that $S^H AS$ is an upper Hessenberg matrix.

(b) Determine a tridiagonal matrix T such that

$$STS^H = \begin{bmatrix} 1 & 0 & 4 & 0 \\ 0 & 3 & 3 & 4 \\ 4 & 3 & 3 & 4 \\ 0 & 4 & 4 & 3 \end{bmatrix}.$$

where S is a product of elementary Hermitian matrices. Choose an appropriate sign for the construction of each elementary Hermitian matrix needed.

Q6. (a) Suppose that the eigenvalue λ_1 of largest modulus and a correspondi[ng]

vector z_1 of an $n \times n$ matrix A have been computed by the Power [method].

Show that there is a non- singular matrix S such that

$$S^{-1}AS = \begin{bmatrix} \lambda & \vdots & b^T \\ \cdots & \cdots & \cdots \\ 0 & \vdots & B \end{bmatrix},$$

where B is an $(n-1) \times (n-1)$ matrix and b is an $(n-1)$- column [vector].

(b) Describe how the other eigenvalues and eigenvectors of A could be [obtained].

(c) It is given that the matrix

$$A = \begin{bmatrix} 2 & 3 & 2 \\ 10 & 3 & 4 \\ 3 & 6 & 1 \end{bmatrix}$$

has an largest eigenvalue 11 with corresponding eigenvector (0.5 ...

Obtain a 2×2 matrix whose eigenvalues are the other eigenvalues [of A].