EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

EXTERNAL DEGREE EXAMINATION IN SCIENCE 2010/2011

SECOND YEAR SECOND SEMESTER (Apr./May., 2017)

EXTMT 204 - RIEMANN INTEGRAL AND SEQUENCES AND SERIES OF FUNCTIONS

(Special Repeat)

Answer All Questions   Time : Two Hours

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1. Let f be a bounded real valued function on $[a, b]$. Explain what is meant by the statement that "f is Riemann integrable over $[a, b]$".

   (a) With the usual notations, prove that a bounded real valued function f on $[a, b]$ is Riemann integrable if and only if for given $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that

   $$U(P, f) - L(P, f) < \varepsilon.$$

   (b) Prove that, if f is continuous on $[a, b]$, then

      i. f is Riemann integrable over $[a, b]$;

      ii. the function $F : [a, b] \to \mathbb{R}$ defined by $F(x) = \int_a^b f(t)\, dt$ is differentiable on $[a, b]$ and $F'(x) = f(x), \ \forall\, x \in [a, b]$.

2. When is an integral $\int_a^b f(x)\, dx$ said to be an improper integral of the first kind, t second kind and the third kind?

What is meant by the statements "an improper integral of the first kind is converge and "an improper integral of the second kind is convergent"?

Discuss the convergence of the improper integral $\int_a^\infty e^{-px}\, dx$.

Test the convergence of the following:

(a) $\int_0^\infty \frac{1}{e^x + 1}\, dx$;

(b) $\int_2^\infty \frac{1}{x-1}\, dx$;

(c) $\int_0^1 \frac{e^x}{\sqrt{x}}\, dx$.

3. (a) What is meant by uniform convergence of a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$.

(b) Prove that the sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ defined on E converges uniforml E if and only if every $\epsilon > 0$ there exists an integer N such that $|f_n(x) - f_m(x)|$ for all $x \in E$ and for all $m, n \geq N$.

(c) Let $\{f_n\}$ be a sequence of functions that are integrable on $[a, b]$ suppose that { converges uniformly on $[a, b]$ to f. Prove that f is integrable and

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx.$$

4.

4. (a) Let $\{f_n\}$ be a sequence of real valued functions defined on $E \subseteq \mathbb{R}$. Suppose that for each $n \in \mathbb{N}$, there is a positive constant M_n such that

$$| f_n(x) | \leq M_n \qquad \text{for all } x \in E,$$

where $\sum_{n=1}^{\infty} M_n$ converges. Prove that $\sum_{n=1}^{\infty} f_n$ converges uniformly on E.

(b) Let $\{f_n\}$, $\{g_n\}$ be two sequence of functions defined over a non-empty set $E \subseteq \mathbb{R}$. Suppose also that

(i) $| S_n(x) | = \left| \sum_{k=1}^{n} f_k(x) \right| \leq M, \quad \forall x \in E, \quad n \in \mathbb{N};$

(ii) $\sum_{k=1}^{\infty} | g_{k+1}(x) - g_k(x) |$ converges uniformly in E;

(iii) $g_n \to 0$ uniformly in E.

Prove that $\sum_{k=1}^{\infty} f_k(x) g_k(x)$ converges uniformly in E.

(c) Show that $\sum_{k=1}^{\infty} \frac{(-1)^k}{k + ax^2}$, $a > 0$ converges uniformly in $\mathbb{R}$.