EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
SECOND EXAMINATION IN SCIENCE - 2009/2010
SECOND SEMESTER (APRIL/MAY, 2012)
MT 204 - RIEMANN INTEGRALS AND SEQUENCE AND SERIES OF FUNCTIONS
(PROPER & REPEAT)

Answer all Questions **Time: Two hours**

---

1. (a) Let $\mathbb{P}$ be a partition of $[a, b]$, where $a, b \in \mathbb{R}$ with $a < b$. If the elements of $\mathbb{P}$ is given by

$$x_i = a\left(\frac{b}{a}\right)^{\frac{i}{n}}, \quad i = 0, 1, \ldots, n,$$

show that

$$||\mathbb{P}|| = b^{1-\frac{1}{n}}\left(b^{\frac{1}{n}} - a^{\frac{1}{n}}\right).$$

[20 marks]

(b) Find the Riemann sum of

$$f(x) = x^2, \quad 1 \leq x \leq 3,$$

corresponding to a partition, $\mathbb{P}$, of points given by

$$t_k = 1 + \frac{2k}{n}, \quad k = 0, 1, 2, \ldots, n,$$

and use it to find $\int_1^3 f(x)\,dx$. [30 marks]

(c) State and prove the necessary and sufficient conditions of Riemann integrability of a real-valued function, and use it to show that every monotone function, $f : [a, b] \longrightarrow \mathbb{R}$, is Riemann Integrable. [50 marks]

2. (a) Check for convergence of the integral

$$\int_1^\infty \frac{1}{x^p}\, dx, \quad p > 0.$$

Hence find the condition on p for which the series

$$\sum_{n=1}^\infty \frac{1}{n^p}$$

is convergent.

[30 marks]

(b) Consider the following statements.

Statement 1: If $0 \le f(x) \le g(x), \forall x \in [a, \infty)$, and if $f(x)$ and $g(x)$ are continuous on $[a, \infty)$, then

$$\int_a^\infty g(x)\, dx \text{ converges} \Longrightarrow \int_a^\infty f(x)\, dx \text{ converges.}$$

Statement 2: Every absolutely convergent integral is convergent.

Statement 3: If $h(x)$ is a monotone function such that $\lim_{x \longrightarrow \infty} h(x) = l \neq \pm\infty$, then

$$\int_a^\infty f(x)\, dx \text{ converges} \Longrightarrow \int_a^\infty f(x)h(x)\, dx \text{ converges.}$$

Using the above statements (1) – (3) to show that

$$\int_1^\infty \frac{\cos x}{x^2} \tan^{-1} x\, dx$$

is convergent.

[30 marks]

(c) Apply μ-test to check whether the following integrals are convergent or not.

i. $\displaystyle\int_0^\infty \frac{1}{x^{1/3}(1+\sqrt{x})}\, dx.$

ii. $\displaystyle\int_0^\infty \frac{x}{(1+x)^3}\, dx.$

[40 marks]

3. (a) Define the terms **Uniform convergence** and **Pointwise convergence** of a sequence of functions. [20 marks]

(b) If $\{f_n\}$ converges uniformly to f on a set S and g is uniformly continuous on a set T containing the ranges of $\{f_n\}$ and f, then prove that $\{g \circ f_n\}$ converges uniformly to $g \circ f$ on S.

What will happen to the above result, if g is not uniformly continuous on T? Justify your answer. [30 marks]

(c) Let $\{f_n\}$ be a sequence of functions that are integrable on $[a, b]$ and suppose that $\{f_n\}$ converges uniformly to f on $[a, b]$. Prove that f is integrable and

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \int_a^b f_n(x)\, dx.$$

[25 marks]

(d) Show that the sequence $\{f_n\}$, where $f_n = nxe^{-nx^2}$, $n = 1, 2, \ldots,$ converges pointwise, but not uniformly on $[0, 1]$. [25 marks]

4. (a) Let S be a subset of a metric space. If $\{f_n\}$ converges uniformly to f on S and each f_n is continuous on $\overline{S}$, where $\overline{S}$ is the closure of S, then prove that $\{f_n\}$ converges uniformly to f on $\overline{S}$. [20 marks]

(b) If $\{f_n\}$ converges uniformly to f on S and if each f_n is continuous at a point x_0 in S, then prove that f is continuous at x_0.

What will happen to the above result if we replace $\{f_n\}$ converges uniformly by just convergence? Justify your answer. [25 marks]

(c) Discuss the uniform convergence of

$$f_n(x) = \frac{x^n}{1 + x^{2n}}$$

in $(-1, 1]$, [2,3] and (-1,0]. [30 marks]

(d) State the **Weierstrass M – test.**

Hence show that $\displaystyle\sum_{n=0}^{\infty} \frac{x^\alpha}{1 + n^2x^2}$ converges uniformly on $(0, 1)$ if $\alpha > 1$.

[25 marks]