EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
SECOND EXAMINATION IN SCIENCE - 2009/2010
SECOND SEMESTER (APRIL/MAY, 2012)
MT 217 - MATHEMATICAL MODELLING
(PROPER & REPEAT)

Answer all Questions — Time: Two hours

Q1. (a) Comment on the following statement.

"Mathematical model building process is an iterative process".

(b) With the aid of a suitable example, explain the mechanistic model based on different levels in the hierarchy?

(c) Types of models play a major role in constructing a mathematical model.

i. What is the important characteristic of a stochastic model?

ii. Differentiate static and dynamic nature of mathematical models in relation to time and types of variables involved.

(d) Aim of a company is to maximize the profit by selling various products produced by the company. For this purpose, a relative quantity of each product has to be determined according to meaningful character of the system. Discuss the nature of this problem in terms of stochastic, static and dynamic notions, and finally state that which of these model characters would be most suited for this problem.

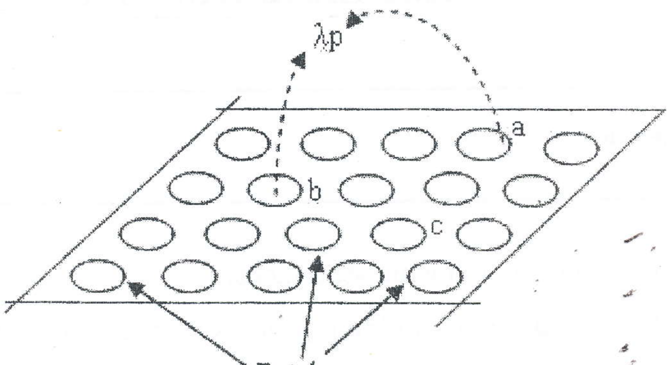
Q2. (a) Flow diagrams are visual aid of describing the system, which give better understanding for modelling more complex systems.

i. Identify the symbols commonly used in flow diagrams.

ii. Using the symbols identified in (i), draw a suitable flow diagram of an energy model for a cattle growth, and write down the constraints of model parameters.

iii. Briefly describe the diagram.

(b) For a population growth, one can be interested in spatial distribution of the population. In this manner, the schematic description given below describes the meta-population.



Explain the spatial and non-spatial distributions of the population by carefully looking at the diagram.

(c) The first car following model can be given by

$$a_{n+1}(t) = v_n(t) - v_{n+1}(t), \quad n = 1, 2, \ldots,$$

where a_n and v_n stand for acceleration and velocity of the n^{th} car, respectively, directly ahead to $(n+1)^{\text{th}}$ car.

i. Does the above model dimensionally correct? Justify your answer.

ii. If not so, re-define the model that has to be in a meaningful form.

iii. Once the driver of $(n+1)^{\text{th}}$ car expects to avoid collision at the rear-end of the n^{th} car, how should he set up his driving?

(Hint: Setting the method of driving is the physical meaning of the above model).

Q3. (a) Define the following terms for the system:

$$\frac{dx}{dt} = f(x,y), \quad \frac{dy}{dt} = g(x,y).$$

i. Critical points.

ii. Trajectory.

(b) In case of two species coexisting, the general solution of the linear system $\dot{\underline{x}} = A\underline{x}$ can be given by

$$\underline{x} = \underline{c}_1 e^{\lambda_1 t} + \underline{c}_2 e^{\lambda_2 t},$$

where $\underline{c}_i$ and λ_i, $i = 1, 2$, are eigen vectors and eigen values, respectively, of the matrix A. Briefly discuss the following terms for a critical point.

i. Stable.

ii. Unstable.

iii. Marginally stable.

(c) The system of differential equations describing the competition between the two species are given by

$$\begin{aligned}\frac{dx}{dt} &= \alpha x - \gamma_{11}x^2 - \gamma_{12}xy, \\ \frac{dy}{dt} &= \beta y - \gamma_{21}xy - \gamma_{22}y^2,\end{aligned}$$

where $x(t)$ and $y(t)$ are the number of first and second species, respectively, present at time t, and α, β and γ_{ij}, $i, j = 1, 2$, are all positive constants.

i. Find critical points of the above system including (0,0).

ii. In case of critical point (0,0), carry out local linearization, and hence obtain the corresponding system of ordinary differential equations.

iii. Find eigen values and state the nature of the critical point (0,0).

iv. Determine eigen vectors and draw the trajectories near (0,0).

Q4. (a) An epidemic is an unusually large, short-tern outbreak of a disease such as AIDS, cholera, etc. A simple model of an epidemic problem can be given by

$$S(t) + I(t) + R(t) = N = \text{constant},$$

where $S(t)$, $I(t)$, and $R(t)$ denote the number of susceptibles, infected persons, and individuals removed from the population by recovery, death, immunization or other means, respectively. Moreover, N is the population size. State the assumptions which led to the construction of the above simple model.

(b) Consider the following model in the absence of removals.

$$\frac{dS}{dt} = -\alpha SI, \quad \frac{dI}{dt} = \alpha SI, \quad \alpha > 0,$$

with initial conditions: $S(0) = S_0$ and $I(0) = I_0$. Show that

$$S(t) = \frac{(N-1)N}{(N-1) + e^{N\alpha t}} \quad \text{and} \quad I(t) = \frac{Ne^{N\alpha t}}{(N-1) + e^{N\alpha t}}.$$

Discuss the limiting behavior of these solutions.

(c) If the given model in (b) is modified by an additional assumption that an infected person recovers and becomes susceptible at a rate proportional to the current number of infectives $I(t)$, with proportionality constant β, show that

$$I(t) = \frac{e^{\kappa t}}{\frac{\alpha}{\kappa}\left[e^{\kappa t} - 1\right] + \frac{1}{I_0}}, \quad \kappa \neq 0,$$

where $\kappa = \alpha N - \beta$. Find the limiting value of $I(t)$.