EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE 2015/2016

FIRST SEMESTER (Nov./Dec., 2017)

PM 201 - VECTOR SPACES AND MATRICES

Answer all questions

Time: Three hours

1. (a) Define a *vector space.*

   (b) Let $V = \{x : x > 0, x \in \mathbb{R}\}$. Define the addition "$\oplus$" and the scalar multiplication "$\odot$" on V as follows:

$$x \oplus y = xy$$

$$\alpha \odot x = x^{\alpha}$$

   $\forall\, x, y \in V$ and $\forall\, \alpha \in \mathbb{R}$. Prove that $(V, \oplus, \odot)$ is a vector space over $\mathbb{R}$.

   (c) Let $A,\ B$ be two subspaces of a vector space V over a field F. Prove the following:

   i. $A + B$ is the smallest subspace of V containing both A and B.

   ii. If $\langle W_1 \rangle = A$ and $\langle W_2 \rangle = B$, then $\langle W_1 \cup W_2 \rangle = A + B$, where $W_1, W_2 \subseteq V$.

2. (a) Let V be an $n-$dimensional vector space. Prove the following:

   (i) A linearly independent set of vectors of V with n elements is a basis for V.

   (ii) Any linearly independent set of vectors of V can be extended as a basis for V.

   (b) Let V be a vector space over a field F.

   i. If $v_1, v_2, \cdots, v_m$ are linearly dependent vectors and $v_1, v_2, \cdots, v_{m-1}$ are linearly independent vectors, then prove that $v_m \in \langle\{v_1, v_2, \cdots, v_{m-1}\}\rangle$.

ii. Let $u_1, u_2, \cdots, u_r$ be linearly independent vectors in V and let $u(\neq 0) \in V$. Prove that $u_i \in \langle\{u, u_1, u_2, \cdots, u_{i-1}\}\rangle$ for some integer i, where $1 \leq i \leq r$ if and only if $u \in \langle\{u_1, u_2, \cdots, u_r\}\rangle$.

iii. Let u_0 and v_0 be linearly independent vectors of V, and let $u_1 = au_0 + bv_0$ and $v_1 = cu_0 + dv_0$, where $a, b, c, d \in F$. Prove that u_1 and v_1 are linearly independent if and only if $ad - bc \neq 0$.

3. (a) State and prove the *dimension* theorem for two subspaces of a finite dimensional vector space.

(b) If L is a subspace of a vector space V, prove that there exists a subspace M of V such that $V = L \oplus M$, where $\oplus$ denotes the direct sum.

(c) i. Let U_1 and U_2 be two subspaces of a vector space V. If $\dim U_1 = 3$, $\dim U_2 = 4$, $\dim V = 6$, show that $U_1 \cap U_2$ contains a non-zero vector.

ii. Let $\mathbb{P}_n = \left\{ \sum_{i=0}^{n} a_i x^i \ : \ a_i \in \mathbb{R}, n \in \mathbb{N} \right\}$ be the set of all polynomials of degree $\leq n$ with real coefficients.

A. If $S = \{2, x, x - x^2, x + x^2\}$ is a subset of $\mathbb{P}_2$, then find the dimension of $\langle S \rangle$.

B. Show that $B = \{1, (x-1), (x-1)^2, (x-1)^3\}$ is a basis of $\mathbb{P}_3$.

4. (a) Define the *range space*, $R(T)$ and the *Null space*, $N(T)$ of a linear transformation T from a vector space V into another vector space W.

(b) Find $R(T)$ and $N(T)$ of a linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, defined by:

$$T(x, y, z) = (x + 2y + 3z, x - y + z, x + 5y + 5z) \ \ \forall \ (x, y, z) \in \mathbb{R}^3.$$

Verify the equation, $\dim V = \dim(R(T)) + \dim(N(T))$ for this linear transformation T, where $V = \mathbb{R}^3$.

(c) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a linear transformation defined by:

$T(x, y, z) = (x + 2y,\ x + y + z,\ z)$ and let $B_1 = \{(1, 1, 1),\ (1, 2, 3),\ (2, -1, 1)\}$ and $B_2 = \{(1, 1, 0),\ (0, 1, 1), (1, 0, 1)\}$ be bases for $\mathbb{R}^3$. Find the following:

(i) The matrix representation of T with respect to the basis B_1;

(ii) The matrix representation of T with respect to the basis B_2 by using the transition matrix.

5. (a) Find the row reduced echelon form of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 3 & 3 & 0 & 2 \\ 2 & 1 & 3 & 3 & -1 & 3 \\ 2 & 1 & 1 & 1 & -2 & 4 \end{pmatrix}.$$

(b) Find the rank of the matrix

$$\begin{pmatrix} 1 & 3 & -2 & 5 & 4 \\ 1 & 4 & 1 & 3 & 5 \\ 1 & 4 & 2 & 4 & 3 \\ 2 & 7 & -3 & 6 & 13 \end{pmatrix}.$$

(c) By applying appropriate row(column) operations, prove that the determinant of the matrix

$$\begin{pmatrix} 1+x_1 & 1 & 1 \\ 1 & 1+x_2 & 1 \\ 1 & 1 & 1+x_3 \end{pmatrix}$$

can be expressed as $x_1x_2x_3\left(1+\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}\right)$, where $x_1, x_2, x_3 \in \mathbb{R}\setminus\{0\}$.

6. (a) State the necessary and sufficient condition for a system of linear equations to be consistent.

Reduce the augmented matrix of the following system of linear equations to its row reduced echelon form and hence determine the conditions on non zero scalars $a_{11}, a_{12}, a_{21}, a_{22}, b_1$ and b_2 such that the system has

(i) a unique solution;

(ii) no solution;

(iii) more than one solution.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2. \end{aligned}$$

(b) Show that the system of equations

$$\begin{aligned} x_1 - 3x_2 + x_3 + \alpha x_4 &= b \\ x_1 - 2x_2 + (\alpha - 1)x_3 - x_4 &= 2 \\ 2x_1 - 5x_2 + (2 - \alpha)x_3 + (\alpha - 1)x_4 &= 3b + 4 \end{aligned}$$

is consistent, for all values of b when $\alpha \neq 1$. Find the value of b for which the system is consistent when $\alpha = 1$ and obtain the general solution for this value.

(c) Prove *Crammer's* rule for 3×3 matrix and use it to solve the following system of linear equations:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 2x_1 + x_2 - 10x_3 &= 4 \\ 2x_1 + 3x_2 - x_3 &= 2. \end{aligned}$$