EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
THIRD YEAR EXAMINATION IN SCIENCE - 2014/2015
SECOND SEMESTER (Dec., 2017/Jan., 2018)
AM 307 - CLASSICAL MECHANICS III

Answer all Questions | Time: Three hours

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1. Two frames of reference S and S' have a common origin O and S' rotates with a constant angular velocity $\underline{\omega}$ relative to S. If a moving particle P has its position vector as $\underline{r}$ relative to O at time t, then show that :

(a) $\dfrac{d\underline{r}}{dt} = \dfrac{\partial \underline{r}}{\partial t} + \underline{\omega} \wedge \underline{r}$;

(b) $\dfrac{d^2\underline{r}}{dt^2} = \dfrac{\partial^2 \underline{r}}{\partial t^2} + 2\underline{\omega} \wedge \dfrac{\partial \underline{r}}{\partial t} + \dfrac{\partial \underline{\omega}}{\partial t} \wedge \underline{r} + \underline{\omega} \wedge (\underline{\omega} \wedge \underline{\dot{r}})$.

An object of mass m initially at rest is dropped to the earth's surface from a height h above the earth's surface. Assuming that the angular speed of the earth about its axis is a constant ω. Prove that after time t the object is deflected east of the vertical by the amount

$$\frac{1}{3}\,\omega g t^3 \cos\lambda\ ,$$

and show that it hits the earth at a point east of the vertical at a distance

$$\frac{2}{3}\,\omega h \cos\lambda \sqrt{\frac{2h}{g}}\ ,$$

where λ is the earth's latitude.

2. (a) Define the following terms:

   i. linear momentum;

   ii. angular momentum;

   iii. moment of force.

   (b) With the usual notations, obtain the equations of motion for a system of N particles in the following form:

   i. $M\underline{f}_G = \sum_{i=1}^{N} \underline{F}_i;$

   (Hint : You may assume $\sum_{i=1}^{N} m_i \underline{v}_i = M\underline{v}_G$.)

   ii. $\frac{d\underline{H}}{dt} = \sum_{i=1}^{N} \underline{r}_i \wedge \underline{F}_i$, where $\sum_{i=1}^{N} \underline{h}_i = \underline{H}$ and $\underline{h}_i = \underline{r}_i \wedge m_i \underline{v}_i$;

   iii. $T = T_G + \frac{1}{2} M v_G^2$.

   A uniform sphere of mass m and radius a is released from rest on a plane inclined at an angle θ to the horizontal. If the sphere rolls down without slipping, show that the acceleration of the center of the sphere is a constant and equal to $\frac{5}{7}\, g \sin\theta$.

3. With the usual notation obtain the Euler's equations for the motion of the rigid body, having a point fixed, in the form:

$$A\dot{\omega}_1 - (B - C)\omega_2\omega_3 = N_1,$$

$$B\dot{\omega}_2 - (C - A)\omega_1\omega_3 = N_2,$$

$$C\dot{\omega}_3 - (A - B)\omega_1\omega_2 = N_3.$$

The principal moments of inertia of a body at the center of mass are A, $3A$, $6A$. The body is so rotated that its angular velocities about the axis are $3n$, $2n$, n respectively. If in the subsequent motion under no forces, ω_1, ω_2, ω_3 denote the angular velocities about the principal axes at time t, show that

$$\omega_1 = 3\omega_3 = \frac{9n}{\sqrt{5}} \sec u \quad \text{and} \quad \omega_2 = 3n \tanh u,$$

where $u = 3nt + \frac{1}{2} \ln 5$.

4. Obtain the Lagrange's equations of motion using D'Alembert's principle for a conservative holonomic dynamical system.

A simple pendulum consists of a mass m_2 attached to a massless rod of length l. A mass m_1 lies at the point of support and can move on a horizontal line lying in the plane in which m_2 moves. Find the Lagrange's equations of motion of the system.

5. With the usual notations, write the Lagrange's equation for the impulsive motion.

Two rods AB and BC, each of length a and mass m, are frictionlessly joined at B and lie on a frictionless horizontal table. Initially both rods are collinear. An impulse I is applied at point A in a direction perpendicular to the line ABC. Prove that, immediately after the application of impulse, the center of mass of AB and BC has the velocity $\left(0, \frac{5I}{4m}\right)$ and $\left(0, -\frac{I}{4m}\right)$ respectively.

6. (a) Define the Poisson bracket.

Show that the Hamiltonian equations of the holonomic system may be written in the form

$$\dot{q}_k = [q_k, H], \quad \dot{p}_k = [p_k, H],$$

and show that for any function $f(q_i, p_i, t)$, $\frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H]$.

(b) i. For a one-dimensional system with the Hamiltonian

$$H = \frac{p^2}{2} - \frac{1}{2q^2},$$

show that there is a constant of the motion

$$D = \frac{pq}{2} - Ht.$$

ii. As a generalization of part (i), for motion in a plane with the Hamiltonian

$$H = |\underline{P}|^n - ar^{-n},$$

where $\underline{P}$ is the vector of the momenta conjugate to the Cartesian coordinates, show that there is a constant of the motion

$$D = \frac{\underline{P} \cdot \underline{r}}{n} - Ht.$$