# EASTERN UNIVERSITY, SRI LANKA
# DEPARTMENT OF MATHEMATICS
# THIRD EXAMINATION IN SCIENCE - 2010/2011
# FIRST SEMESTER(April, 2013)
# MT 302 - COMPLEX ANALYSIS
# (PROPER/REPEAT)

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**Answer all Questions** **Time: Three hours**

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Q1. (a) Let $A \subseteq \mathbb{C}$ be an open set and let $f : A \to \mathbb{C}$. Define what is meant by $f$ being **analytic** at $z_0 \in A$.

(b) Let the function$f(z) = u(x, y) + iv(x, y)$ be defined throughout some $\epsilon$ neighborhood of a point $z_0 = x_0 + y_0$. Suppose that the first - order partial derivatives of the functions $u$ and $v$ with respect to $x$ and $y$ exist everywhere in that neighborhood and that they are continuous at $(x_0, y_0)$. Prove that, if those partial derivatives satisfy the **Cauchy-Riemann** equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

at $(x_0, y_0)$, then the derivative $f'(z_0)$ exists.

(c) (i) Prove that $u = e^{-x}(x \sin y - y \cos y)$ is harmonic.

(ii) Find the function $v(x, y)$ such that $f(z) = u(x, y) + iv(x, y)$ is analytic.

Q2. (a) (i) Define what is meant by a curve $\gamma : [\alpha, \beta] \to \mathbb{C}$ smooth.

(ii) For a path $\gamma$ and a continuous function $f : \gamma \to \mathbb{C}$, define $\int_\gamma f(z)dz$.

(b) Prove that if $\gamma$ is a path and $f \in C(\gamma)$, then $|f(z)| \leq M$, for all $z \in \gamma$ and $M \geq 0$ such that $\left|\int_\gamma f(z)dz\right| \leq ML$, where $L$ =length$(\gamma)$.

(c) State the **Cauchy's Integral Formula**.

By using the **Cauchy's Integral Formula** compute the following integrals:

(i) $\displaystyle\int_{C(0;2)} \frac{z}{(9-z^2)(z+i)}dz;$

(ii) $\displaystyle\int_{C(0;1)} \frac{1}{(z-a)^k(z-b)}dz$, where $k \in \mathbb{Z}$, $|a| > 1$ and $|b| < 1$.

Q3. (a) State and prove the **Mean Value Property for Analytic Functions**.

(b) (i) Define what is meant by the function $f : \mathbb{C} \to \mathbb{C}$ being **entire**.

(ii) Let $f$ be analytic on $D(z_0; r)$ and $0 < s < r$. Let

$$M(s) = \sup\{|f(z)| : |z - z_0| = s\}.$$

Prove that

$$|f^{(n)}(z_0)| \leq \frac{n!M(s)}{s^n},$$

and if $f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$, then

$$|a_n| \leq \frac{M(s)}{s^n}.$$

(c) Prove the **Maximum - Modulus Theorem**: Let $f$ be analytic in an open connected set $A$. Let $\gamma$ be a simple closed path that is, contained together with its inside in $A$. Let

$$M := \sup_{z\in\gamma} |f(z)|.$$

If there exists $z_0$ inside $\gamma$ such that $|f(z_0)| = M$, then $f$ is constant throughout $A$. Consequently, if $f$ is not constant in $A$, then

$$|f(z)| < M, \ \forall z \text{ inside } \gamma.$$

(State any theorem you use without proof)

Q4. (a) Let $\delta > 0$ and let $f : D^*(z_0; \delta) \to \mathbb{C}$, where $D^*(z_0; \delta) := \{z : 0 < |z - z_0| < \delta\}$. Define what is meant by

(i) $f$ having a singularity at $z_0$;

(ii) the order of $f$ at $z_0$;

(iii) $f$ having a pole or zero at $z_0$ of order $m$;

(iv) $f$ having a simple pole or simple zero at $z_0$.

(b) Prove that an isolated singularity $z_0$ of $f$ is removable if and only if $f$ is bounded on some deleted neighborhood $D^*(z_0;\delta)$ of $z_0$.

(c) Prove that if $f$ has a simple pole at $z_0$, then

$$Res(f;z_0) = \lim_{z \longrightarrow z_0}(z - z_0)f(z_0).$$

Q5. Let $f$ be analytic in the upper - half plane $\{z : Im(z) \geq 0\}$, except at finitely many points, none on the real axis. Suppose there exist $M, R > 0$ and $\alpha >.1$ such that

$$|f(z)| \leq \frac{M}{|z|^\alpha}, |z| \geq R \text{ with } \mathrm{Im(z)} \geq 0.$$

Then prove that

$$I := \int_{-\infty}^{\infty} f(x)dx$$

converges (exists) and

$$I = 2\pi i \times \text{Sum of Residues of f in the upper half plane.}$$

Hence evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{1+x^2}dx.$$

(You may assume without proof the Residue Theorem).

Q6. (a) State the **Argument Theorem**.

(b) Prove **Rouche's Theorem** : Let $\gamma$ be a simple closed path in an open starset $A$. Suppose that

(i) $f, g$ are analytic in $A$ except for finitely many poles, none lying on $\gamma$.

(ii) $f$ and $f+g$ have finitely many zeros in $A$.

(iii) $|g(z)| < |f(z)|, z \in \gamma$. Then

$$ZP(f+g;\gamma) = ZP(f;\gamma)$$

where $ZP(f+g;\gamma)$ and $ZP(f;\gamma)$ denote the number of zeros - number of poles inside $\gamma$ of $f+g$ and $f$ respectively, where each is counted as many times as its order.

(c) State the **Fundamental theorem of Algebra**.

(d) Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.