EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

THIRD EXAMINATION IN SCIENCE - 2015/2016

SECOND SEMESTER (Oct./Nov., 2018)

PM 301 - GROUP THEORY

Answer all questions Time : Three hours

1. (a) Show that the set

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid ad \neq 0 \right\}$$

forms a group under the matrix multiplication, where $a, b, d \in \mathbb{Q}$, the set of rational numbers.

(b) An element a is called an *idempotent element* if $a * a = a$. Prove that a group with binary operation $*$ has exactly one idempotent element.

(c) If $a^2 = e$ for all elements a in a group G, then show that G is abelian.

(d) Let a and b are commutative elements of a group G. Using the mathematical induction or otherwise, prove that $(ab)^n = a^n b^n$ for each positive integer n.

2. (a) Prove that a nonempty subset H of a group G is a subgroup of G if and only if

i. $a, b \in H$ implies that $ab \in H$,

ii. $a \in H$ implies that $a^{-1} \in H$.

(b) Let G be a group and a be a fixed element of G. Prove the following:

i. the centralizer

$$C(a) = \{g \in G \mid ga = ag\}$$

is a subgroup of G,

ii. For any $a \in G$, $C(a) = C(a^{-1})$.

3. (a) Prove that every cyclic group is abelian.
   (b) Find the orders of each subgroup of the cyclic group $\mathbb{Z}_{24}$. List every generator of subgroups of order 6 in $\mathbb{Z}_{24}$.
   (c) Let G be a group and define a map $\lambda_g : G \longrightarrow G$ by $\lambda_g(a) = ga$. Prove that λ_g permutation of G.
   (d) Express the following permutations of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ as a product of disj cycles and then as a product of transpositions.
      i. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$
      ii. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$

4. (a) Let $\phi : G \to G'$ be a homomorphism between the groups G and G'. Prove that is the identity element of G, then $\phi(e)$ is the identity element of G'.
   (b) Which of the following maps are homomorphisms? If the map is a homomorph find it's kernel.
      i. $\phi : \mathbb{R}^* \longrightarrow GL_2(\mathbb{R})$ define by $\phi(a) = \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$,
      ii. $\phi : \mathbb{R} \longrightarrow GL_2(\mathbb{R})$ define by $\phi(a) = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$,

      where $\mathbb{R}^* = \mathbb{R} - \{0\}$ and $GL_2(\mathbb{R})$ is a group of 2×2 matrices in $\mathbb{R}$.
   (c) If $\phi : G \to H$ is a group homomorphism and G is abelian, prove that $\phi(G)$ is abelian.

5. (a) Let H be a subgroup of a group G and $g \in G$. Prove that $Hg = H$ if and o $g \in H$.
   (b) If H and K are subgroups of a group G and $g \in G$, show that $g(H \cap K) = gH$
   (c) State the *Lagrange's theorem.* Using the Lagrange's theorem or otherwise fin index of the following subgroups:
      i. the index of $\langle 3 \rangle$ in $\mathbb{Z}_{24}$,
      ii. the index of $\langle 18 \rangle$ in $\mathbb{Z}_{36}$.
   (d) Find the partition of Z_{12} into cosets of the subgroup $\langle 2 \rangle$.

6. (a) Let H be a subgroup of a group G. Prove that if G is abelian, then G/H is abelian.

   (b) Let T be the group of nonsingular upper triangular 2×2 matrices with entries in $\mathbb{R}$; that is, matrices of the form

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix},$$

where $a, b, c \in \mathbb{R}$ and $ac \neq 0$. Let U consists of matrices of the form

$$\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix},$$

where $x \in \mathbb{R}$. Prove the following:

i. U is a subgroup of T,

ii. U is abelian,

iii. U is normal in T,

iv. T/U is abelian.