EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

THIRD EXAMINATION IN SCIENCE - 2014/2015

SECOND SEMESTER (Oct./Nov., 2018)

PM 303 - FUNCTIONAL ANALYSIS-I

Answer all questions | Time : Two hours

1. (a) Let X be a vector space of all ordered pairs $x = (\zeta_1, \zeta_2)$ of real numbers. Show that
$$\|x\| = |\zeta_1| + |\zeta_2|$$
defines a norm on X.

   (b) If $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ are normed spaces, show that with the usual operations the product vector space $X = X_1 \times X_2$ is a normed space with the norm defined by
$$\|x\| = \max(\|x_1\|_1, \|x_2\|_2)$$
where $x = (x_1, x_2) \in X$.

   (c) If d is a metric induced by a norm on a normed linear space X, then prove that
      i. $d(x + a, y + a) = d(x, y)$
      ii. $d(\alpha x, \alpha y) = |\alpha| \, d(x, y)$

   for all $x, y, a \in X$ and any scalar α.

2. (a) Define a *Cauchy sequence* in a normed linear space.

   (b) Prove that on a finite dimensional normed linear space, any norm $\|\cdot\|$ is equivalent to any other norm $\|\cdot\|_0$.

   (c) If $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent norms on a normed linear space X, then show that (x_n) is a Cauchy sequence in $(X, \|\cdot\|)$ if and only if (x_n) is a Cauchy sequence $(X, \|\cdot\|_0)$.

3. (a) Let X and Y be normed linear spaces and let $T : D(T) \longrightarrow Y$ be a linear operator from the domain $D(T) \subseteq X$ of T to Y. If the range of T, $R(T) \subseteq Y$, then prove that the inverse operator $T^{-1} : R(T) \longrightarrow D(T)$ exists if and only if $Tx = 0$ implies that $x = 0$.

(b) Prove that if T^{-1} exists, then it is a linear operator.

(c) Let T be a bounded linear operator from a normed linear space X onto a normed linear space Y. If there is a positive number b such that

$$||Tx|| \geq b\,||x|| \quad \forall x \in X,$$

then show that $T^{-1} : Y \longrightarrow X$ exists and bounded.

4. (a) Define the *sublinear functional* on a vector space.

i. Show that a sublinear functional p satisfies $p(0) = 0$ and $p(-x) \geq -p(x)$.

ii. Show that a norm on a vector space X is a sublinear functional on X.

(b) Let X and Y be normed linear spaces and let $S, T \in B(X, Y)$, the space of bounded linear operators from X to Y, with $||Sx|| \leq k_1||x||$ and $||Tx|| \leq k_2||x||$ for all $x \in X$. Prove the following:

i. $||(S + T)x|| \leq (k_1 + k_2)||x||$ for all $x \in X$,

ii. $||(\lambda S)x|| \leq |\lambda|k_1||x||$ for all $x \in X$ and for any scalar λ,

iii. $B(X, Y)$ is a vector space with respect to the operations defined by

$$(T + S)(x) = Tx + Sx \quad \text{for all } x \in X,$$

$$(\alpha T)(x) = \alpha Tx \quad \text{for all } x \in X \text{ and for any scalar } \alpha.$$