EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

XTERNAL DEGREE EXAMINATION IN SCIENCE - 2008/2009

FIRST YEAR SECOND SEMESTER (Jun/Sep., 2015)

EXTMT 102 - REAL ANALYSIS

Repeat

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nswer all Questions | Time: Three hours

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1. (a) i. Define the terms 'Supremum' and 'Infimum' of a non-empty subset of $\mathbb{R}$.

ii. State the completeness property of $\mathbb{R}$, and use it to prove that every non-empty bounded below subset of $\mathbb{R}$ has an infimum.

(b) i. Prove that an upper bound u of a non-empty bounded above subset S of $\mathbb{R}$ is the supremum of S if and only if for every $\epsilon > 0$, there exist an $x \in S$ such that $x > u - \epsilon$.

ii. Let S be non-empty subset of $\mathbb{R}$ that is bounded above and $a \in \mathbb{R}$. Let the set $a + S$ be defined as

$$a + S = \{a + x : x \in S\}.$$

Prove that $\text{Sup}(a + S) = a + \text{Sup } S$. $\mathbb{R}$.

(c) Find the Supremum and Infimum of the set

$$S = \left\{1 - \frac{1}{n};\ n \in \mathbb{N}\right\}.$$

Q2. (a) State what it means by a sequence of real numbers (x_n) converges to a limit a(a

(b) Prove that every convergent sequence of real numbers is bounded.

(c) State the Monotone Convergent Theorem.

Let a sequence (x_n) be defined by

$a - x_{n+1} = \sqrt{a^2 - x_n}$ for all $n \geq 1$ and $0 < x_1 < a^2$, where $a > 1$.Prove that

i. $0 < (x_n) < a^2$ for all $n \in \mathbb{N}$;

ii. x_n is strictly decreasing sequence ;

iii. $\lim_{n \longrightarrow \infty} (x_n) = 0$.

Q3. (a) Define the following terms:

i. a subsequence of a sequence;

ii. Cauchy sequence.

(b) State and prove the Bozano-Weierstrass theorem.

(c) Prove that a sequence (x_n) of real numbers is cauchy if and only if it is con-vergent .

(d) Let (a_n) and (b_n) be two Cauchy sequences and $c_n = |a_n - b_n|$. Show that $(c$ is a Cauchy sequence.

Q4. (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Explain what is meant by the function f has limit $l(\in \mathbb{R})$ at a point $a(\in \mathbb{R})$.

(b) If $\lim_{x \longrightarrow a} f(x) = l$, then show that $\lim_{x \longrightarrow a} |f(x)| = |l|$. Is the converse true? Just your answer.

(c) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function and $\lim_{x \longrightarrow a} f(x) = l(\neq 0)$.

Prove the following:

i. there exist $\delta > 0$ such that $\frac{|l|}{2} < |f(x)| < \frac{3|l|}{2}$, for all x such that $0 < |x - a| < \delta$;

ii. $\lim_{x \longrightarrow a} \frac{1}{f(x)} = \frac{1}{l}$, if $f(x) \neq 0$, $\forall x \in \mathbb{R}$.

(a) Define what it means to say that a real-valued function f is continuous at a point $'a'$ in its domain.

Let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0; \\ 1 & \text{if } x = 0. \end{cases}$$

Prove that, f is continuous at $x = 0$.

(b) Prove that if a function $f : [a,b] \longrightarrow \mathbb{R}$ is continuous on $[a,b]$, then it is bounded on $[a,b]$.

(c) State the Intermediate Value Theorem and use it to show that the equation $2x^2(x+2)-1 = 0$ has a root in each of the intervals $(-2,-1)$, $(-1,0)$ and $(0,1)$.

. (a) i. Define what it means to say that the real-valued function f is differentiable at a point $'a'$ in its domain.

ii. Prove that every differentiable function is continuous. Is the converse true? Justify your answer.

(b) State the Mean-Value Theorem and use it to prove

$$x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}}, \quad \forall x \in (0,1).$$

(c) Suppose that f and g are continuous on $[a,b]$ differentiable on (a,b) and $g'(x) \neq 0$ for all $x \in (a,b)$. Prove that there exists $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

If $f(d) = g(d) = 0$ for some $d \in (a,b)$ deduce that

$$\lim_{x \to d} \frac{f(x)}{g(x)} = \lim_{x \to d} \frac{f'(x)}{g'(x)}.$$