

Mixed potentials in radiative stellar collapse

S. Thirukkanesh¹ and S. D. Maharaj²
Astrophysics and Cosmology Research Unit,
School of Mathematical Sciences,
University of KwaZulu-Natal,
Private Bag X54001,
Durban 4000,
South Africa.

Abstract

We study the behaviour of a radiating star when the interior expanding, shearing fluid particles are traveling in geodesic motion. We demonstrate that it is possible to obtain new classes of exact solutions in terms of elementary functions without assuming a separable form for the gravitational potentials or initially fixing the temporal evolution of the model unlike earlier treatments. A systematic approach enables us to write the junction condition as a Riccati equation which under particular conditions may be transformed into a separable equation. New classes of solutions are generated which allow for mixed spatial and temporal dependence in the metric functions. We regain particular models found previously from our general classes of solutions.

1 Introduction

In an astrophysical environment, it is likely that a star emits radiation and particles in the process of gravitational collapse. In this situation, the heat flow in the interior of a star should not be neglected; the interior spacetime of the collapsing radiating star should match to the exterior spacetime described by the Vaidya¹ solution. Exact models of relativistic radiating stars are important for the investigation of the cosmic censorship hypothesis and gravitational collapse^{2,3}. Santos⁴ formulated the junction conditions for shear-free collapse, matching the interior metric with the exterior Vaidya metric at the boundary of the star, which made it possible to generate exact models.

¹Permanent address: Department of Mathematics, Eastern University, Chenkalady, Sri Lanka.

Electronic mail: thirukkanesh@yahoo.co.uk

²Electronic mail: maharaj@ukzn.ac.za

This treatment enabled us to investigate physical features such as surface luminosity, dynamical stability, relaxation effects, particle production at the surface and temperature profiles for radiating stars in general relativity. De Oliviera *et al*⁵ proposed a radiating model of an initial interior static configuration leading to slow gravitational collapse. It had been shown earlier that the slowest possible collapse arises in the case of shear-free fluid interiors⁶. In a recent treatment Herrera *et al*⁷ proposed a relativistic radiating model with a vanishing Weyl tensor, in a first order approximation, without solving the junction condition exactly. Then Maharaj and Govender⁸ and Herrera *et al*⁹ solved the relevant junction condition exactly, and generated classes of solutions in terms of elementary functions which contain the Friedmann dust solution as special case. Later Mithry *et al*¹⁰ obtained several other classes of solution by transforming the junction condition to the form of an Abel equation of the first kind. These exact models have proved to be useful in analysing the relativistic behaviour of a collapsing objects in the stellar scenario.

Another useful approach in studying the effects of dissipation is due to Kolassis *et al*¹¹ in which the fluid particles are restricted to travel along geodesics. In the absence of heat flow, the interior Friedmann dust solution is regained. This particular exact solution formed the basis for many investigations involving physical features such as the rate of collapse, surface luminosity and temperature profiles. The physical investigations include the analytic model of radiating gravitational collapse in a spherical geometry with neutrino flux by Grammenos and Kolassis¹², the model describing realistic astrophysical processes with heat flux by Tomimura and Nunes¹³, and models undergoing gravitational collapse with heat flow which serves as a possible mechanism for gamma-ray bursts by Zhe *et al*¹⁴. Herrera *et al*¹⁵ investigated geodesic fluid spheres in coordinates which are not comoving in the presence of anisotropic pressures. Govender *et al*¹⁶ demonstrated that the temperature in casual thermodynamics for particles traveling on geodesics produces higher central values than the Eckart theory. The first exact solution, that we are aware of, with nonzero shear was obtained by Naidu *et al*¹⁷, considering geodesic motion of fluid particles; later Rajah and Maharaj¹⁸ obtained two classes of nonsingular solutions by solving a Riccati equation. Note that the geodesic condition arises in several other astrophysical situations including Euclidean stars, with the real and proper radius equal, in the absence of dissipation¹⁹.

In this paper we attempt to extend the previous treatments of Naidu *et al*¹⁷ and Rajah and Maharaj¹⁸ through a systematic approach by studying the fundamental junction condition. Our intention is to show that the nonlinear boundary condition may be analysed mathematically to produce an infinite classes of exact solutions. In Section 2, we present the geodesic model governing the description of a radiating star using the Einstein field equations together with the junction conditions in the presence

of anisotropic pressure. We show that it is possible to transform the junction condition into a separable equation by placing restrictions on one of the gravitational potentials. Two new classes of solutions are obtained in terms of arbitrary functions of the radial coordinate and we regain the models found in the past for particular choices of arbitrary functions in Section 3. In Section 4, we discuss some physical aspects of the models generated.

2 The model

In the context of general relativity, the form for the interior space time of a spherically symmetric collapsing star with nonzero shear when the fluid trajectories are geodesics is given by the line metric

$$ds^2 = -dt^2 + B^2 dr^2 + Y^2(d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

where B and Y are functions of both the temporal coordinate t and radial coordinate r . The fluid four-velocity vector \mathbf{u} is given by $u^a = \delta_0^a$ which is comoving. For the line element (1), the four-acceleration \dot{u}^a , the expansion scalar Θ and the magnitude of the shear scalar σ are given by

$$\dot{u}^a = 0, \quad (2a)$$

$$\Theta = \frac{\dot{B}}{B} + 2\frac{\dot{Y}}{Y}, \quad (2b)$$

$$\sigma = \frac{1}{3} \left(\frac{\dot{Y}}{Y} - \frac{\dot{B}}{B} \right), \quad (2c)$$

respectively, and dots denote the differentiation with respect to t . The energy momentum tensor for the interior matter distribution is described by

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + \pi_{ab} + q_a u_b + q_b u_a, \quad (3)$$

where p is the isotropic pressure, ρ is the energy density of the fluid, π_{ab} is the stress tensor and q_a is the heat flux vector. Anisotropy plays a significant role in gravitational collapse and affects the mass, luminosity and stability of relativistic spheres; these features have been highlighted in the treatments of Chan²⁰ and Herrera and Santos²¹. The stress tensor has the form

$$\pi_{ab} = (p_r - p_t) \left(n_a n_b - \frac{1}{3} h_{ab} \right), \quad (4)$$

where p_r is the radial pressure, p_t is the tangential pressure and \mathbf{n} is a unit radial vector given by $n^a = \frac{1}{B} \delta_1^a$. The isotropic pressure is given by

$$p = \frac{1}{3}(p_r + 2p_t) \quad (5)$$

in terms of the radial pressure and the tangential pressure. For the line element (1) and matter distribution (3) the Einstein field equations become

$$\rho = 2\frac{\dot{B}\dot{Y}}{BY} + \frac{1}{Y^2} + \frac{\dot{Y}^2}{Y^2} - \frac{1}{B^2} \left(2\frac{Y''}{Y} + \frac{Y'^2}{Y^2} - 2\frac{B'Y'}{BY} \right), \quad (6a)$$

$$p_r = -2\frac{\ddot{Y}}{Y} - \frac{\dot{Y}^2}{Y^2} - \frac{1}{Y^2} + \frac{1}{B^2} \frac{Y'^2}{Y^2}, \quad (6b)$$

$$p_t = -\left(\frac{\ddot{B}}{B} + \frac{\dot{B}\dot{Y}}{BY} + \frac{\ddot{Y}}{Y} \right) + \frac{1}{B^2} \left(\frac{Y''}{Y} - \frac{B'Y'}{BY} \right), \quad (6c)$$

$$q = -\frac{2}{B^2} \left(\frac{\dot{B}Y'}{BY} - \frac{\dot{Y}'}{Y} \right), \quad (6d)$$

where the heat flux $q^a = (0, q, 0, 0)$ is radially directed and primes denote the differentiation with respect to r . The system of equations (6a)-(6d) governs the most general situation in describing geodesic matter distributions in a spherically symmetric gravitational field. These equations describe the gravitational interaction of a shearing matter distribution with heat flux and anisotropic pressure for particles traveling along geodesics. From (6a)-(6d), we observe that if the gravitational potentials $B(t, r)$ and $Y(t, r)$ are specified then the expressions for the matter variables ρ, p_r, p_t and q follow by simple substitution.

The Vaidya exterior spacetime of radiating star is given by

$$ds^2 = -\left(1 - \frac{2m(v)}{R}\right) dv^2 - 2dv dR + R^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (7)$$

where $m(v)$ denotes the mass of the fluid as measured by an observer at infinity. The line element (7) is utilized to describe incoherent null radiation which flows in the radial direction relative to the hypersurface Σ which denotes the boundary of the star. The matching of the interior spacetime (1) with the exterior spacetime (7) generate the set of junction conditions on the hypersurface Σ given by

$$dt = \left(1 - \frac{2m}{R_\Sigma} + 2\frac{dR_\Sigma}{dv}\right)^{\frac{1}{2}} dv, \quad (8a)$$

$$Y(R_\Sigma, t) = R_\Sigma(v), \quad (8b)$$

$$m(v)_\Sigma = \left[\frac{Y}{2} \left(1 + \dot{Y}^2 - \frac{Y'^2}{B^2}\right) \right]_\Sigma, \quad (8c)$$

$$(p_r)_\Sigma = (qB)_\Sigma. \quad (8d)$$

The nonvanishing of the radial pressure at the boundary Σ is reflected in equation (8d). Equation (8d) is an additional constraint which has to be satisfied together with the system of equations (6a)-(6d).

The junction condition (8d) in the case of shear-free spacetimes was first derived by Santos⁴, and later it was extended by Glass²² to incorporate spacetimes with nonzero shear. On substituting (6b) and (6d) in (8d) we obtain

$$2Y\ddot{Y} + \dot{Y}^2 - \frac{Y'^2}{B^2} + \frac{2}{B}Y\dot{Y}' - 2\frac{\dot{B}}{B^2}YY' + 1 = 0 \quad (9)$$

which has to be satisfied on Σ . Equation (9) governs the gravitational behaviour of the radiating anisotropic star with nonzero shear and no acceleration. As equation (9) is highly nonlinear, it is difficult to solve without some simplifying assumption. This equation comprises two unknown functions $B(t, r)$ and $Y(t, r)$. To generate a solution we have to specify one of the functions so that the resulting equation is tractable.

3 New exact solutions

For convenience we rewrite equation (9) in the form of the Riccati equation in the gravitational potential B as follows

$$\dot{B} = \left[\frac{\ddot{Y}}{Y'} + \frac{\dot{Y}^2}{2YY'} + \frac{1}{2YY'} \right] B^2 + \frac{\dot{Y}'}{Y'}B - \frac{Y'}{2Y}. \quad (10)$$

Equation (10) was analyzed by Nogueira and Chan²³ who obtained approximate solutions using numerical techniques. To properly describe the physical features of a radiating relativistic star exact solutions are necessary, preferably written in terms of elementary functions. An exact solution was found by Naidu *et al*¹⁷ which was singular at the stellar centre. A new class of solutions was established by Rajah and Maharaj¹⁸, which contain the Naidu *et al*¹⁷ models, in which the singularities at the centre were shown to be avoidable.

The Riccati equation (10), which has to be satisfied on the stellar boundary Σ , is highly nonlinear and difficult to solve. Rajah and Maharaj¹⁸ obtained solutions in an *ad hoc* fashion by assuming that the gravitational potential $Y(t, r)$ is a separable function and specifying the temporal evolution of the model. In this paper we demonstrate that it is possible to find new exact solutions systematically without assuming separable forms for $Y(t, r)$ and not fixing the temporal evolution of the model *a priori*. If we introduce the transformation

$$B = ZY' \quad (11)$$

then equation (10) becomes

$$\dot{Z} = \frac{1}{2Y} [FZ^2 - 1], \quad (12)$$

where we have set

$$F = 2Y\ddot{Y} + \dot{Y}^2 + 1.$$

We observe that equation (12) becomes a separable equation in Z and t , and therefore integrable, if we let F be a constant or a function of r only. In other word, (12) is integrable as long as F is independent of t . We emphasize that in this approach we have not made any assumption about the separability of the metric coefficients $B(t, r)$ and $Y(t, r)$ or restricted the t -dependence. We demonstrate that this approach leads to two new classes of solutions in the following sections.

3.1 Analytic solution I

If we set $F = 1$ then the function Y is given by

$$Y(r, t) = [R_1(r)t + R_2(r)]^{2/3}, \quad (13)$$

where $R_1(r)$ and $R_2(r)$ are arbitrary functions of r . For this case equation (12) becomes

$$\dot{Z} = \frac{1}{2[R_1(r)t + R_2(r)]^{2/3}} [Z^2 - 1]. \quad (14)$$

On integrating (14) we obtain

$$Z = \frac{1 + f(r) \exp [3(R_1t + R_2)^{1/3}/R_1]}{1 - f(r) \exp [3(R_1t + R_2)^{1/3}/R_1]}, \quad (15)$$

where $f(r)$ is a function of integration. Hence from (11), (13) and (15) we get

$$B = \frac{2}{3} \left[\frac{1 + f(r) \exp [3(R_1t + R_2)^{1/3}/R_1]}{1 - f(r) \exp [3(R_1t + R_2)^{1/3}/R_1]} \right] \frac{[R_1't + R_2']}{[R_1t + R_2]^{1/3}}. \quad (16)$$

Therefore the line element (1) takes the particular form

$$ds^2 = -dt^2 + \frac{4}{9} \left[\frac{1 + f(r) \exp [3(R_1t + R_2)^{1/3}/R_1]}{1 - f(r) \exp [3(R_1t + R_2)^{1/3}/R_1]} \right]^2 \frac{[R_1't + R_2']^2}{[R_1t + R_2]^{2/3}} dr^2 \\ + [R_1(r)t + R_2(r)]^{4/3} (d\theta^2 + \sin^2 \theta d\phi^2), \quad (17)$$

The line element (17) is given in terms of arbitrary functions $R_1(r)$, $R_2(r)$ and $f(r)$ so that it is possible to generate infinite number of solutions for different choices of these functions. Observe that if we set

$$R_1 = 0, R_2 = r^{3/2}$$

then the equivalent of (17) is

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

which is the flat Minkowski spacetime. As the curvature then vanishes we require $R_1 \neq 0$ and the t -dependence in Y is maintained.

It is interesting to see that for particular forms of the arbitrary functions we regain models found previously. If we set

$$R_1 = R^{3/2}, \quad R_2 = aR^{3/2}$$

then the line element (17) reduces to

$$ds^2 = -dt^2 + (t+a)^{4/3} \left\{ R^2 \left[\frac{1 + f(r) \exp [3(t+a)^{1/3}/R]}{1 - f(r) \exp [3(t+a)^{1/3}/R]} \right]^2 dr^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\}. \quad (18)$$

The line element (18) corresponds to the first category of the Rajah and Maharaj¹⁸ models for an anisotropic radiating star with shear. Furthermore note that if we set $a = 0$ and $R = r$ then (18) reduces to

$$ds^2 = -dt^2 + t^{4/3} \left\{ \left[\frac{1 + f(r) \exp [3t^{1/3}/r]}{1 - f(r) \exp [3t^{1/3}/r]} \right]^2 dr^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\}. \quad (19)$$

The metric (19) was first found by Naidu *et al*¹⁷ in their analysis of pressure anisotropy and heat dissipation in a spherically symmetric radiating star undergoing gravitational collapse. Note that when

$$R_1 = r^{3/2}, \quad R_2 = 0, \quad f(r) = 0$$

the line element (17) takes on the simple form

$$ds^2 = -dt^2 + t^{4/3} [dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)]. \quad (20)$$

The line element (20) corresponds to the Friedmann metric when the fluid is in the form of dust with vanishing heat flux.

3.2 Analytic solution II

If we set $F = 1 + R_1^2(r)$ then the function Y is given by

$$Y(r, t) = R_1(r)t + R_2(r), \quad (21)$$

where $R_1(r)$ and $R_2(r)$ are functions of r only. For this case equation (12) becomes

$$\dot{Z} = \frac{[R_1^2 + 1]}{2[R_1 t + R_2]} \left[Z^2 - \frac{1}{[R_1^2 + 1]} \right]. \quad (22)$$

The solution of (22) can be written as

$$Z = \frac{1}{\sqrt{R_1^2 + 1}} \left[\frac{1 + g(r)[R_1 t + R_2] \sqrt{R_1^2 + 1}/R_1}{1 - g(r)[R_1 t + R_2] \sqrt{R_1^2 + 1}/R_1} \right], \quad (23)$$

where $g(r)$ is the function of integration. Hence from (11), (21) and (23) we get

$$B = \frac{1}{\sqrt{R_1^2 + 1}} \left[\frac{1 + g(r)[R_1 t + R_2] \sqrt{R_1^2 + 1}/R_1}{1 - g(r)[R_1 t + R_2] \sqrt{R_1^2 + 1}/R_1} \right] [R_1' t + R_2']. \quad (24)$$

Therefore the line element (1) takes the particular form

$$ds^2 = -dt^2 + \frac{1}{[R_1^2 + 1]} \left[\frac{1 + g(r)[R_1 t + R_2] \sqrt{R_1^2 + 1}/R_1}{1 - g(r)[R_1 t + R_2] \sqrt{R_1^2 + 1}/R_1} \right]^2 [R_1' t + R_2']^2 dr^2 + [R_1(r)t + R_2(r)]^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (25)$$

in terms of arbitrary functions $R_1(r)$, $R_2(r)$ and $g(r)$. Therefore it is again possible to generate an infinite number of solutions to (9). As in §3.1 we cannot set $R_1 = 0$ because the Minkowski spacetime is results.

The new class of solutions given above does contain previously known models. Note that when

$$R_1 = R, \quad R_2 = aR$$

the line element (25) reduces to

$$ds^2 = -dt^2 + (t+a)^2 \left\{ \frac{R'^2}{[R^2 + 1]} \left[\frac{1 + h(r)[t + a] \sqrt{R^2 + 1}/R}{1 - h(r)[t + a] \sqrt{R^2 + 1}/R} \right]^2 dr^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\}, \quad (26)$$

where we have defined the new arbitrary function $h(r) = g(r)R^{\sqrt{R^2 + 1}/R}$. The line element (26) corresponds to the second category of the Rajah and Maharaj¹⁸ anisotropic radiating stars with shear. Such solutions are difficult to interpret but could play a role in gravitational collapse for strong fields.

4 Discussion

The simple form of the exact solutions that have been generated in our treatment make it possible to study the physical features of the model such as luminosity, rate of collapse, particle production, neutrino flux and temperature profiles. In particular explicit forms for the causal temperature can be found utilizing the Maxwell-Cattaneo heat transport equation

$$\tau h_a^b \dot{q}_b + q_a = -\kappa (h_a^b \nabla_b T + T u_a). \quad (27)$$

This has been done for special cases by Naidu *et al*¹⁷ and Rajah and Maharaj¹⁸. The causal temperature is well behaved in the stellar interior, in particular the causal temperature is everywhere greater than the acausal temperature. Other choices of the

metric functions in our new classes of solutions (17) and (25) generate similar behaviour and highlight the role of inhomogeneity in dissipative processes. Consequently our new general class of exact radiating stars is physically reasonable.

We have generated a new class of stellar models with shear and expansion in geodesic motion. All solutions found previously arise as special cases in our treatment. Previous analysis were *ad hoc* and effectively required that one of the metric functions should be a separable function. The resulting Riccati equation could then be solved. In this work, we did not assume separability of the metric functions and did not initially fix the temporal evolution of the model which is different from earlier treatments of this problem. Our more general approach enabled us to solve the Riccati equation systematically; we found that the Riccati equation could be transformed to a separable equation. Two classes of exact solutions were explicitly demonstrated to the junction condition. These were shown to contain the Naidu *et al*¹⁷ and Rajah and Maharaj¹⁸ metrics. The fundamental reason that new solutions are possible is that we have relaxed the condition of separability in the metric function $Y(t, r)$.

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