Radiating relativistic matter in geodesic motion

S. Thirukkanesh[1] **and S. D. Maharaj**[2]
Astrophysics and Cosmology Research Unit,
School of Mathematical Sciences,
University of KwaZulu-Natal,
Private Bag X54001,
Durban 4000,
South Africa.

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Abstract

We study the gravitational behaviour of a spherically symmetric radiating star when the fluid particles are in geodesic motion. We transform the governing equation into a simpler form which allows for a general analytic treatment. We find that Bernoulli, Riccati and confluent hypergeometric equations are possible. These admit solutions in terms of elementary functions and special functions. Particular models contain the Minkowski spacetime and the Friedmann dust spacetime as limiting cases. Our infinite family of solutions contains specific models found previously. For a particular metric we briefly investigate the physical features, derive the temperature profiles and plot the behaviour of the casual and acasual temperatures.

1 Introduction

Relativistic models of radiating stars are useful in the investigation of cosmic censorship hypothesis, gravitational collapse with dissipation, formation of superdense matter, dynamical stability of radiating matter and temperature profiles in the context of irreversible thermodynamics. The general model, incorporating all necessary physical requirements and variables, is complicated and difficult to solve; the treatments of Herrera *et al* [1] and Di Prisco *et al* [2] involving physically meaningful charged spherically symmetric collapse with shear and dissipation illustrate the complexity of the processes. To solve the field equations, and to find tractable forms for the gravitational and matter variables, we need to make simplifying assumptions. De Oliviera

[1]Permanent address: Department of Mathematics, Eastern University, Chenkalady, Sri Lanka.
[2]Electronic mail: maharaj@ukzn.ac.za

et al [3] proposed a radiating model in which an initial static configuration leads to collapse. This approach may be adapted to describe the end state of collapse as shown by Govender *et al* [4]. In a recent treatment Herrera *et al* [5] proposed a model in which the form of Weyl tensor was highlighted when studying radiative collapse with an approximate solution. Maharaj and Govender [6], Herrera *et al* [7] and Misthry *et al* [8] showed that it is possible to solve the field equations and boundary conditions exactly in this scenario. For recent treatments involving collapse with equations of state and formation of black holes see Goswami and Joshi [9], [10].

A useful approach in understanding the effects of dissipation is due to Kolassis *et al* [11] in which the fluid trajectories are assumed to be geodesic. In the limit, in the absence of heat flow, the interior Friedmann dust solution was regained. This solution formed the basis for many investigations involving the physical behaviour such as the rate of collapse, surface luminosity and temperature profiles. These include the analytic model of radiating spherical gravitational collapse with neutrino flux by Grammenos and Kolassis [12], the model describing realistic astrophysical processes with heat flow by Tomimura and Nunes [13], and models undergoing collapse with heat flow as a possible mechanism for gamma-ray bursts by Zhe *et al* [14]. Herrera *et al* [15] considered geodesic fluid spheres in coordinates which are not comoving but with anisotropic pressures. Govender *et al* [16] showed that the behaviour of the temperature in casual thermodynamics for geodesic motion produces higher central temperatures than the Eckart theory. The first exact solution with shear, satisfying the boundary conditions, was obtained by Naidu *et al* [17] by considering geodesic fluid trajectories. Later Rajah and Maharaj [18] extended this treatment and obtained classes of models which are nonsingular at the centre.

It is clear that the assumption of geodesic motion is physically acceptable and has been used by other investigators in attempts to describe realistic astrophysical processes. In this paper we attempt to perform a systematic treatment on the governing equation at the boundary for shear-free collapse by assuming the geodesic motion of the fluid particle. Our intention is to show that the nonlinear boundary condition may be analysed systematically to produce an infinite family of exact solutions. In Section 2, we present the model governing the description of a radiating star using the Einstein field equations together with the junction conditions. We show that it is possible to transform the junction condition to a Bernoulli equation and a Riccati equation. Solutions are obtained in terms of elementary functions in Section 3. In Section 4, we show that the boundary condition, under relevant assumptions, can be written in the form of a confluent hypergeometric equation. We demonstrate that an infinite family of solutions in terms of elementary functions are possible. In Section 5, we obtain the explicit form for the causal temperature using the truncated form of the

Maxwell-Cattaneo heat transport equation for a particular metric. This illustrates that the simple forms for the gravitational potentials obtained in this paper are physically plausible. Some concluding statements are made in Section 6.

2 The model

We analyse a spherically symmetric relativistic radiating star undergoing shear-free gravitational collapse. This assumption is reasonable when modelling a radiating star in relativistic astrophysics. If we suppose that the particle trajectories are geodesic then the acceleration vanishes. Then the line element, for the matter distribution interior to the boundary of the radiating star, is given by

$$ds^2 = -dt^2 + B^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{1}$$

where $B = B(r,t)$ is the only surviving metric function. The energy momentum tensor including radiation for the interior spacetime is given by

$$T_{ab} = (\rho + p)u_a u_b + p g_{ab} + q_a u_b + q_b u_a \tag{2}$$

where the energy density ρ, the pressure p and the heat flow vector $\mathbf{q}$ are measured relative to the timelike fluid 4-velocity $u^a = \delta^a_0$. The heat flow vector takes the form $q^a = (0, q, 0, 0)$ since $\mathbf{q}\cdot\mathbf{u} = 0$ for heat flow which is radially directed.

The nonzero components of Einstein field equations, for the line element (1) and the energy momentum tensor (2), can be written as

$$\rho = 3\frac{\dot{B}^2}{B^2} - \frac{1}{B^2}\left(2\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{4}{r}\frac{B'}{B}\right), \tag{3a}$$

$$p = -2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + \frac{1}{B^2}\left(\frac{B'^2}{B^2} + \frac{2}{r}\frac{B'}{B}\right), \tag{3b}$$

$$p = -2\frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} + \frac{1}{B^2}\left(\frac{B''}{B} - \frac{B'^2}{B^2} + \frac{1}{r}\frac{B'}{B}\right), \tag{3c}$$

$$q = -\frac{2}{B^2}\left(-\frac{\dot{B}'}{B} + \frac{B'\dot{B}}{B^2}\right), \tag{3d}$$

where dots and primes denote differentiation with respect to time t and r respectively. Equating (3b) and (3c) we obtain the condition

$$\left(\frac{1}{B}\right)'' = \frac{1}{r}\left(\frac{1}{B}\right)' \tag{4}$$

which is the condition of pressure isotropy. Equation (4) is integrable and we obtain

$$B = \frac{d}{C_2(t) - C_1(t)r^2} \tag{5}$$

where $C_1(t)$ and $C_2(t)$ are functions of time, and d is a constant. As the functional form for the potential B is specified the matter variables ρ, p and q are known quantities, and the system (3) has been solved in principle.

The interior spacetime (1) has to be matched across the boundary $r = b$ to the exterior Vaidya spacetime

$$ds^2 = -\left(1 - \frac{2m(v)}{R}\right) dv^2 - 2dvdR + R^2(d\theta^2 + \sin^2\theta d\phi), \tag{6}$$

where $m(v)$ denotes the mass of the star as measured by an observer at infinity. The hypersurface at the boundary is denoted by Σ. The matching of the line elements (1) and (6), and matching of the extrinsic curvature at the surface of the star, leads to a set of equations. The boundary conditions at Σ have the form

$$\begin{aligned} dt &= \left[\left(1 - \frac{2m}{R} + 2\frac{dR}{dv}\right)^{1/2} dv\right]_\Sigma, & \text{(7a)}\\ (rB)_\Sigma &= R_\Sigma, & \text{(7b)}\\ p_\Sigma &= (qB)_\Sigma, & \text{(7c)}\\ [m(v)]_\Sigma &= \left[\frac{r^3}{2}\left(\dot{B}^2 B - \frac{B'^2}{B}\right) - r^2 B'\right]_\Sigma, & \text{(7d)} \end{aligned}$$

where the subscript means that the relevant quantities are evaluated on Σ.

From (3), (5) and (7c) we generate the condition

$$\begin{aligned} &-4db(\dot{C}_1 C_2 - C_1\dot{C}_2)(C_1 b^2 - C_2) - 4C_1C_2(C_1b^2 - C_2)^2 \\ &\qquad -2d^2(\ddot{C}_1 b^2 - \ddot{C}_2)(C_1b^2 - C_2) + 5d^2(\dot{C}_1 b^2 - \dot{C}_2)^2 = 0. \end{aligned} \tag{8}$$

Effectively (8) results from the nonvanishing of the pressure gradient across the hypersurface Σ. Equation (8) governs the dynamical evolution of shear-free radiating stars in which fluid trajectories are geodesic. To complete the description in this particular radiating model we need to explicitly solve the differential equation (8).

3 Generating Analytic Solutions

A particular solution to (8) was found by Kolassis *et al* (1988) by inspection. We show that it is possible to transform (8) into familiar differential equations which admit solutions in closed form. Our method is a more systematic approach in solving equation (8). In this approach we let

$$C_1 b^2 - C_2 = u(t). \tag{9}$$

On substituting (9) into (8) we can write

$$4bdu^2\dot{C}_1 + 4(u^2 - bd\dot{u})uC_1 - 4b^2u^2C_1^2 = d^2(2u\ddot{u} - 5\dot{u}^2). \tag{10}$$

Equation (10) is simpler than (8) and can be viewed as a first order differential equation in the variable C_1. In general, (10) is a Riccati equation (in C_1), and is difficult solve in the above form without simplifying assumptions. For the integration of (10), in terms of elementary functions, we consider the following two cases:

3.1 Bernoulli equation

We set

$$2u\ddot{u} - 5\dot{u}^2 = 0 \tag{11}$$

so that the function u is given by

$$u = \alpha \quad \text{or} \quad u = \beta(t+\gamma)^{-2/3}, \tag{12}$$

where α, β and γ are real constants. With the assumption (11), (10) becomes

$$4bdu^2\dot{C}_1 + 4(u^2 - bd\dot{u})uC_1 - 4b^2u^2C_1^2 = 0. \tag{13}$$

Equation (13) is nonlinear but is a Bernoulli equation which can be linearised in general.

When $u = \alpha$, equation (13) becomes

$$\dot{C}_1 + \frac{\alpha}{bd}C_1 - \frac{b}{d}C_1^2 = 0 \tag{14}$$

which is a Bernoulli equation with constant coefficients. The solution of (14) is given by

$$C_1 = \frac{\alpha}{b^2 - \exp\left(\frac{\alpha(t+e)}{bd}\right)},$$

where e is the constant of integration. Consequently the remaining function C_2 has the form

$$C_2 = \frac{\alpha\exp\left(\frac{\alpha(t+e)}{bd}\right)}{b^2 - \exp\left(\frac{\alpha(t+e)}{bd}\right)}.$$

Hence the interior line element (1) has the specific form

$$ds^2 = -dt^2 + \frac{d^2}{\alpha^2}\left[\frac{b^2 - \exp\left(\frac{\alpha(t+e)}{bd}\right)}{r^2 - \exp\left(\frac{\alpha(t+e)}{bd}\right)}\right]^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{15}$$

in terms of exponential functions. We believe that this is a new solution to the Einstein field equations for a radiating star. It is interesting to observe that if we set $\alpha = d$

when $t \to \infty$ (or large values of the constant e) then (15) becomes the flat Minkowski spacetime

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

which is a limiting case.

When $u = \beta(t+\gamma)^{-2/3}$, (13) becomes

$$\dot{C}_1 + \left[\frac{\beta}{bd}(t+\gamma)^{-2/3} + \frac{2}{3}(t+\gamma)^{-1}\right] C_1 - \frac{b}{d}C_1^2 = 0 \tag{16}$$

which is also a Bernoulli equation with variable coefficients. The solution of (16) is given by

$$C_1 = \frac{\beta}{\left[b^2 + \beta f \exp\left(\frac{3\beta(t+\gamma)^{1/3}}{bd}\right)\right]}(t+\gamma)^{-2/3}$$

where f is the constant of integration. Consequently the remaining function C_2 is given by

$$C_2 = \frac{-\beta^2 f \exp\left(\frac{3\beta(t+\gamma)^{1/3}}{bd}\right)}{\left[b^2 + \beta f \exp\left(\frac{3\beta(t+\gamma)^{1/3}}{bd}\right)\right]}(t+\gamma)^{-2/3}.$$

Hence the interior line element (1) takes the particular form

$$ds^2 = -dt^2 + \frac{d^2}{\beta^2}\left[\frac{b^2 + \beta f \exp\left(\frac{3\beta(t+\gamma)^{1/3}}{bd}\right)}{r^2 + \beta f \exp\left(\frac{3\beta(t+\gamma)^{1/3}}{bd}\right)}\right]^2 (t+\gamma)^{4/3}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{17}$$

If we set

$$\gamma = 0, d = \left(\frac{M}{6}\right)^{1/3} b, f = \frac{3}{ab^2}, \beta = -\frac{b^2}{3}$$

then (17) becomes

$$ds^2 = -dt^2 + \frac{9\left(\frac{M}{6}\right)^{2/3}}{b^2}\left[\frac{1 - ab^2 \exp\left(\frac{6t}{M}\right)^{1/3}}{1 - ar^2 \exp\left(\frac{6t}{M}\right)^{1/3}}\right]^2 t^{4/3}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

which was first found by Kolassis *et al* [11]. Here we have shown that their model found by inspection arises naturally as a solution of a Bernoulli equation. It is easy to see that for large values of the constant f we obtain

$$ds^2 = -dt^2 + t^{4/3}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]$$

from (17). This corresponds to the Friedmann metric when the fluid is in the form of dust with vanishing heat flux.

3.2 Riccati equation

If we set

$$u^2 - db\dot{u} = 0 \tag{18}$$

then the function u is given by

$$u = -bd(t+a)^{-1} \tag{19}$$

where a is a constant. In this case equation (10) becomes

$$4bd\dot{C}_1 - 4b^2C_1^2 + d^2(t+a)^{-2} = 0, \tag{20}$$

which is an inhomogeneous Riccati equation. The solution of equation (20) has the form

$$C_1 = \frac{-d\left[1-\sqrt{2}+(1+\sqrt{2})g(t+a)^{\sqrt{2}}\right]}{2b\left[1+g(t+a)^{\sqrt{2}}\right]}(t+a)^{-1}, \tag{21}$$

where g is the constant of integration. Consequently the remaining function has the form

$$C_2 = bd\left\{1 - \frac{\left[1-\sqrt{2}+(1+\sqrt{2})g(t+a)^{\sqrt{2}}\right]}{2\left[1+g(t+a)^{\sqrt{2}}\right]}\right\}(t+a)^{-1}.$$

Hence the interior metric (1) has the specific form

$$ds^2 = -dt^2 + \frac{d^2(t+a)^2}{\left[C_1(r^2-b^2)(t+a)-bd\right]^2}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{22}$$

which is written in terms of C_1. We believe that (22) is a new solution for a radiating star whose particles are constrained to travel on geodesics. The simple form of (22) will assist in studying the physical features of our model. The solution (22) arises in a natural way once we realise that the underlying dynamical equation (8) at the boundary is a Riccati equation.

4 Special functions

The solutions found in the previous sections all have power law forms for the quantity u. It is possible that other solutions in terms of elementary functions or special functions may exist with a power law representation for u. Consequently in this section we attempt to generate a general class of solutions to the model (8) by assuming

$$u = \alpha(t+a)^n. \tag{23}$$

On substituting (23) into (10) we obtain

$$(t+a)^2\dot{C}_1 + \left[\frac{\alpha}{db}(t+a)^{n+1} - n\right](t+a)C_1 - \frac{b}{d}(t+a)^2 C_1^2 = -\frac{d}{4b}n(3n+2). \qquad (24)$$

The nonlinear equation (24) is a Riccati equation and it is difficult to solve the equation in the above form. If we introduce a transformation

$$\frac{b}{d}C_1 = -\frac{\dot{U}}{U} \qquad (25)$$

then (24) becomes the second order linear differential equation

$$(t+a)^2\ddot{U} + \left[\frac{\alpha}{db}(t+a)^{n+1} - n\right](t+a)\dot{U} - \frac{n(3n+2)}{4}U = 0 \qquad (26)$$

in the function U with variable coefficients. We can transform (26) to simpler form if we let

$$\psi = (t+a)^{n+1},\ W = U\psi^{-k},\ k = \frac{(n+1) \pm \sqrt{4n(n+1)+1}}{2(n+1)}. \qquad (27)$$

Then (26) becomes

$$(n+1)\psi\frac{d^2W}{d\psi^2} + \left[\frac{\alpha}{bd}\psi + 2k(n+1)\right]\frac{dW}{d\psi} + \frac{\alpha k}{bd}W = 0. \qquad (28)$$

If we let

$$X = \frac{-\alpha\psi}{bd(n+1)},\ Y(X) = W(\psi)$$

then (28) has the equivalent form

$$X\frac{d^2Y}{dX^2} + (2k - X)\frac{dY}{dX} - kY = 0. \qquad (29)$$

Observe that (29) is the confluent hypergeometric equation with solution in terms of special functions in general.

Note that the solution of (29) can be written in terms of

$$\begin{aligned} Y &= \mathcal{J}(k, 2k; X), \\ W &= \mathcal{J}\left(k, 2k; \frac{-\alpha\psi}{bd(n+1)}\right) \end{aligned}$$

where $\mathcal{J}$ are Kummer functions. In general the solution of the equation (28) can be written in terms of the Kummer series. Observe that when $k > 0$ we can write

$$\begin{aligned} \tilde{W} &= \mathcal{J}(k, 2k; X) \\ &= \frac{\Gamma(2k)}{[\Gamma(k)]^2}\int_0^1 e^{X\tau}\left[\tau(1-\tau)\right]^{k-1} d\tau \end{aligned} \qquad (30)$$

as a particular solution of the differential equation (28) where $\Gamma(z) = \int_0^\infty e^{-\tau}\tau^{z-1}d\tau$ is the gamma function. From (30) we note that the solution can be expressed in terms of elementary functions for all natural numbers k. Consequently the differential equation (24) admits solutions in terms of elementary functions when k is a natural number.

4.1 Particular metrics

We can regain previous cases from the general form (30). We illustrate this feature for particular values of k. When $k = 1$, we obtain $n = 0$ or $n = -2/3$. For this case the particular solution of the equation (28) becomes

$$\tilde{W} = \frac{e^X - 1}{X}, \quad X = \frac{-\alpha\psi}{bd(n+1)} \tag{31}$$

with the help of (30).

When $n = 0$, from (27) and (31) we can easily see that

$$\tilde{U} = \frac{bd}{\alpha}\left[1 - \exp\left(\frac{-\alpha(t+a)}{bd}\right)\right]$$

is a particular solution of the equation (26). Then with the help of (25) we find that

$$\tilde{C_1} = \frac{\alpha}{b^2\left[1 - \exp\left(\frac{\alpha(t+a)}{bd}\right)\right]} \tag{32}$$

is a particular solution of (24) which is given by

$$\dot{C_1} + \frac{\alpha}{bd}C_1 - \frac{b}{d}{C_1}^2 = 0. \tag{33}$$

The general solution of (33) becomes

$$C_1 = \frac{\alpha D}{b^2\left[D + \exp\left(\frac{\alpha(t+a)}{bd}\right)\right]},$$

where D is an arbitrary constant. Consequently the interior metric (1) has the specific form

$$ds^2 = -dt^2 + \frac{b^4 d^2}{\alpha^2}\left[\frac{D + \exp\left(\frac{\alpha(t+a)}{bd}\right)}{Dr^2 + b^2\exp\left(\frac{\alpha(t+a)}{bd}\right)}\right]^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{34}$$

in terms of exponential functions. Note that the line element (34) reduces to the metric (15) if we set $D = -b^2$.

When $n = -2/3$, from (27) and (31) we observe that

$$\tilde{U} = \frac{bd}{3\alpha}\left[1 - \exp\left(\frac{-3\alpha(t+a)^{1/3}}{bd}\right)\right]$$

is a particular solution of the equation (26). Hence with the help of (25) we obtain

$$\tilde{C_1} = \frac{\alpha(t+a)^{2/3}}{b^2\left[1 - \exp\left(\frac{3\alpha(t+a)^{1/3}}{bd}\right)\right]} \tag{35}$$

as a particular solution of (24) which has the form

$$\dot{C}_1 + \left[\frac{\alpha}{bd}(t+a)^{-2/3} + \frac{2}{3}(t+a)^{-1}\right] C_1 - \frac{b}{d}{C_1}^2 = 0. \tag{36}$$

The general solution of (36) becomes

$$C_1 = \frac{\alpha D(t+a)^{2/3}}{b^2\left[D + \exp\left(\frac{3\alpha(t+a)^{1/3}}{bd}\right)\right]},$$

where D is an arbitrary constant. Consequently the interior metric (1) takes the particular form

$$ds^2 = -dt^2 + \frac{b^4 d^2}{\alpha^2}\left[\frac{D + \exp\left(\frac{3\alpha(t+a)^{1/3}}{bd}\right)}{Dr^2 + \exp\left(\frac{3\alpha(t+a)^{1/3}}{bd}\right)}\right]^2 (t+a)^{4/3}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{37}$$

Note that the line element (37) reduces to the line element (17) if we set $D = \frac{b^2}{\alpha f}$.

4.2 A new solution

It is possible to generate an infinite family of new solutions from the general form (30) by specifying values for the parameter k. These may correspond to new solutions for a radiating sphere which are not accelerating. We illustrate this process by taking $k = 2$ (so that $n = -2$ or $n = -4/5$) in (30). We consider only the case $n = -2$ as the integration procedure is same for other values of k (or n) . For this case the particular solution of the equation (28) becomes

$$\tilde{W} = \frac{6}{X^3}\left[2 + X + (X-2)e^X\right], \quad X = \frac{-\alpha\psi}{bd(n+1)}. \tag{38}$$

When $n = -2$, from (27) and (38) we observe that

$$\tilde{U} = \frac{6b^2d^2}{\alpha^3}\left[[2bd(t+a)+\alpha] - [2bd(t+a)-\alpha]\exp\left(\frac{\alpha}{bd(t+a)}\right)\right]$$

is a particular solution of the equation (26). Hence with the help of (25) we obtain

$$\tilde{C_1} = \frac{2b^2d^2(t+a)^2 - [2bd(t+a)(bd(t+a)-\alpha)+\alpha^2]\exp\left(\frac{\alpha}{bd(t+a)}\right)}{b^2\left[(2bd(t+a)-\alpha)\exp\left(\frac{\alpha}{bd(t+a)}\right) - (2bd(t+a)+\alpha)\right](t+a)^2} \tag{39}$$

is a particular solution of (24) which has the form

$$\dot{C}_1 + \left[\frac{\alpha}{bd}(t+a)^{-2} + 2(t+a)^{-1}\right] C_1 - \frac{b}{d}{C_1}^2 = -\frac{2d}{b}(t+a)^{-2}. \tag{40}$$

The general solution of (40) becomes

$$C_1 = -\frac{\left[[\alpha^2 + 2bd(t+a)(bd(t+a)-\alpha)]\exp\left(\frac{\alpha}{bd(t+a)}\right) + 2b^2d^2D(t+a)^2\right]}{b^2\left[D(2bd(t+a)+\alpha) + (2bd(t+a)-\alpha)\exp\left(\frac{\alpha}{bd(t+a)}\right)\right](t+a)^2}, \qquad (41)$$

where D is an arbitrary constant. Consequently the interior metric (1) has the specific form

$$ds^2 = -dt^2 + \frac{d^2}{[C_1(r^2-b^2) + \alpha(t+a)^{-2}]^2}\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \qquad (42)$$

where C_1 is given by (41). Hence we have found a new solution to the boundary condition (8) by specifying a particular value for the parameter k. This process can be repeated for other values of k and an infinite family of solutions are possible in which the gravitational potentials can be expressed in terms of elementary functions.

5 Physical analysis

The simple forms of the gravitational potentials found in this paper permit a detailed study of the physical features of a radiating star. In this study we consider the particular line element (34) and set $\alpha = bd$ and $a = 0$ to obtain

$$ds^2 = -dt^2 + b^2\left[\frac{D+\exp(t)}{Dr^2 + b^2\exp(t)}\right]^2\left[dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \qquad (43)$$

for simplicity. For the metric (43) the matter variables can be written as

$$\rho = \frac{3D\exp(t)\left\{\exp(t)\left[Db^4 + 6b^2Dr^2 + Dr^4 + 4b^4\exp(t)\right] + 4D^2r^4\right\}}{[D+\exp(t)]^2\left[Dr^2 + b^2\exp(t)\right]^2}, \qquad (44a)$$

$$\begin{aligned} p = {} & \frac{D\exp(t)}{[D+\exp(t)]^2\left[Dr^2 + b^2\exp(t)\right]^2}\times \\ & \left\{\exp(t))\left[2b^2\exp(t)(r^2-3b^2) - 2Db^2r^2 - 3D(r^4+b^4)\right] + 2D^2r^2(b^2-3r^2)\right\}, \end{aligned} \qquad (44b)$$

$$q = \frac{4Dr\exp(t)}{[D+\exp(t)]^2}. \qquad (44c)$$

When $D = 0$ then (43) becomes the Minkowski metric with $\rho = p = q = 0$. The matter variables are expressed in simple analytic forms which facilitate the analysis of the physical behaviour. From (44) we have that at the centre of the sphere

$$\begin{aligned} \rho_0 &= \frac{3D\left[D + 4\exp(t)\right]}{[D+\exp(t)]^2} \\ p_0 &= -\frac{3D[D+2\exp(t)]}{[D+\exp(t)]^2} \\ q_0 &= 0 \end{aligned}$$

so that ρ_0 and p_0 have finite values at the centre $r = 0$ with vanishing heat flux q_0. The gravitational potentials in (43) are finite at the centre and nonsingular in the stellar interior. The quantities ρ, p and q are well behaved and regular in the interior of the sphere, at least in regions close to the centre. At later times as $t \to \infty$ we note that $q \propto r$ so that the magnitude of the heat flux depends linearly on the radial coordinate.

Next we briefly consider the relativistic effect of casual temperature of this model. The Maxwell-Cattaneo heat transport equation, in the absence of rotation and viscous stresses, is given by

$$\tau h_a{}^b \dot{q}_b + q_a = -\kappa \left(h_a{}^b \nabla_b T + T \dot{u}_a \right), \tag{45}$$

where $h_{ab} = g_{ab} + u_a u_b$ projects into the comoving rest space, T is the local equilibrium temperature, κ (≥ 0) is the thermal conductivity and τ (≥ 0) is the relaxation time. Equation (45) reduces to the acausal Fourier heat transport equation when $\tau = 0$. For the line element (1), the casual transport equation (45) can be written as

$$T(t,r) = -\frac{1}{\kappa} \int \left[\tau (q\dot{B}) B + q B^2 \right] dr \tag{46}$$

for geodesic motion. Martinez [19], Govender *et al* [16] and Di Prisco *et al* [20] have demonstrated that the relaxation time τ on the thermal evolution, plays a significant role in the latter stages of collapse. For the line element (43), (46) becomes

$$\begin{aligned} T(t,r) \;\; &= \;\; \frac{\tau b^2 \exp(t) \left\{ 2D^2 r^2 - b^2 \exp(t) \left[\exp(t) - D \right] \right\}}{\kappa \left[\exp(t) + D \right] \left[b^2 \exp(t) + D r^2 \right]^2} \\ &\quad + \frac{2 b^2 \exp(t)}{\kappa \left[b^2 \exp(t) + D r^2 \right]} + h(t), \end{aligned} \tag{47}$$

where $h(t)$ is a function of integration. For simplicity we assumed that τ and κ are constant. The function $h(t)$ may be related to the central temperature $T_c(t)$ by

$$h(t) = T_c(t) - \frac{\tau \left[D - \exp(t) \right]}{\kappa \left[D + \exp(t) \right]} - \frac{2}{\kappa}. \tag{48}$$

From (47) and (48) the temperature can be written as

$$\begin{aligned} T(t,r) \;\; &= \;\; T_c(t) - \frac{\tau D r^2 \left\{ D r^2 \left[D - \exp(t) \right] - 2 b^2 \exp(t) \right\}}{\kappa \left[D + \exp(t) \right] \left[D r^2 + b^2 \exp(t) \right]^2} \\ &\quad - \frac{2 D r^2}{\kappa \left[D r^2 + b^2 \exp(t) \right]}. \end{aligned} \tag{49}$$

When $\tau = 0$, we can regain the acausal (Eckart) temperature profiles from (49). In Fig. 1, we plot the casual (solid line) and acasual (dashed line) temperatures against the radial coordinate on the interval $0 \leq r \leq 5$ for particular parameter values ($\kappa = \tau = 1, b = 5, D = 70$ and $h(t) = 0$) on the spacelike hypersurface $t = 1$. We observe that the temperature is monotonically decreasing from centre to the boundary in both casual and acasual cases. It is clear that the casual temperature is greater than the acasual temperature throughout the stellar interior. At the boundary Σ we have

$$T(t, r_\Sigma)_{\text{casual}} \simeq T(t, r_\Sigma)_{\text{acasual}}.$$

Our figures have been generated by assuming constant values for the parameters τ and κ. Changing the values of the relaxation time and the thermal conductivity would produce different gradients for the curves but the result would not change qualitatively.

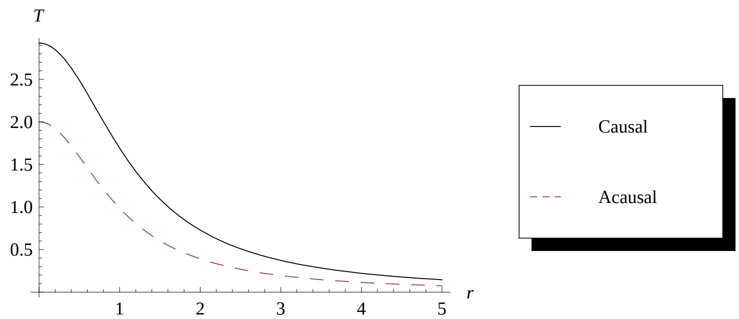

Figure 1: Temperature T vs radial coordinate r ($\tau = 1$).

6 Discussion

It is possible to introduce shear in geodesic motion as shown by Naidu *et al* [17] and Rajah and Maharaj [18] in the description of a radiating star. The solutions that follow are governed by a Riccati equation and have a complicated form. Consequently in this paper we have considered the simpler case of a shear-free metric with particles traveling on geodesic trajectories. The master equation, governing the boundary condition of the stellar model, was transformed to a simpler form. Under certain assumptions a Bernoulli equation is possible. This Bernoulli equation admits two solutions in terms of elementary functions: the first solution contains the Minkowski spacetime as a limiting case and the second solution corresponds to the Kolassis *et al* [11] model with the Friedmann dust spacetime as the limiting case. A general class of solutions are possible if we transform the master equation to a confluent hypergeometric equation. The resulting transformed equation admits solution in terms of special functions namely the Kummer functions. By specifying particular values for a parameter in the special function we demonstrate that an infinite family of solutions, in terms of elementary functions, are possible. The simple form of the solutions makes it possible to study the physical features of the model and to find an analytic form for the causal temperature.

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