EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST YEAR EXAMINATION IN SCIENCE - 2013/2014

SECOND SEMESTER - (APRIL/MAY, 2016)

AM 104 - DIFFERENTIAL EQUATIONS AND FOURIER SERIES

(PROPER & REPEAT)

Answer **All** Questions Time Allowed: **3** Hours

Q1. (a) State the necessary and sufficient condition for the ordinary differential equation (ODE)

$$M(x,y)\,dx + N(x,y)\,dy = 0 \tag{1}$$

to be *exact*. **[10 Marks]**

If

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} + N\,\mu(x)$$

then show that $e^{\int \mu(x)\,dx}$ is an integrating factor of the ODE (1).

[20 Marks]

Using the above result or otherwise find the general solution of the following ODE

$$x(x+y+1)\frac{dy}{dx} + y^2 + 3xy + 2y = 0.$$

[40 Marks]

(b) Find the general solution of the nonlinear first-order Riccati equation given by

$$x^2\frac{dy}{dx} + 2 - 2xy + x^2y^2 = 0.$$

[30 Marks]

Q2. Let $D \equiv d/dx$ be a differential operator. Show that a particular integra[l] of the ODE

$$(D-a)(D-b)y = P(x),$$

where a, b are arbitrary real constants and $P(x)$ is an arbitrary functio[n] in its variable, is given by

$$y = e^{ax} \int e^{(b-a)x} \left(\int P e^{-bx}\, dx \right) dx.$$

[40 Marks

Using the above result or otherwise, obtain the general solution of th[e] following ODE:

(i) $(D^2 - 3D + 2)y = \sin e^{-x}$;

(ii) $(D^2 - 1)y = (1 + e^{-x})^{-2}$.

[60 Marks

Q3. (a) Let $x = e^t$. Show that

$$x\frac{d}{dx} \equiv \mathcal{D}, \quad x^2 \frac{d^2}{dx^2} \equiv \mathcal{D}^2 - \mathcal{D},$$

and

$$x^3 \frac{d^3}{dx^3} \equiv \mathcal{D}(\mathcal{D}-1)(\mathcal{D}-2),$$

where $\mathcal{D} \equiv \dfrac{d}{dt}$. [20 Marks

Use the above results to find the general solution of the followin[g] Cauchy-Euler differential equation

$$(x^3 D^3 + xD - 1)y = 3x^4,$$

where $D \equiv \dfrac{d}{dx}$. [40 Marks]

(b) Find the general solution of the following system of linear ODEs

$$(D^2 - 2)x - 3y = e^{2t},$$
$$(D^2 + 2)y + x = 0.$$

[40 Marks

Q4. (a) Define what is meant by the point, $x = x_0$, being

(i) an *ordinary* ;

(ii) a *singular*;

(iii) a *regular singular*

point of the ODE

$$y'' + p(x)y' + q(x)y = 0,$$

where the prime denotes differentiation with respect to x, and $p(x)$ and $q(x)$ are rational functions.

[30 Marks]

(b) (i) Find the regular singular point(s) of the ODE

$$xy'' + (x-1)y' - 2y = 0. \qquad (2)$$

(ii) Use the method of Frobenius to find the general solution of the equation (2).

[70 Marks]

Q5. (a) Solve the following system of ODEs:

(i) $$\frac{dx}{x(2y^4 - z^4)} = \frac{dy}{y(z^4 - 2x^4)} = \frac{dz}{z(x^4 - y^4)};$$

(ii) $$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{y+z} = \frac{dz}{y-z}.$$

[30 Marks]

(b) Write down the condition of integrability of the total differential equation

$$P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = 0.$$

[5 Marks]

Hence solve the following equation

$$z(2x^3 - z)\,dx + 2x^2yz\,dy + x(x+z)\,dz = 0.$$

[15 Marks]

(c) Find the equation of the integral surface satisfying the quasi-linear partial differential equation (PDE),

$$yp - 2xyq = 2xz$$

which contains the curve $x = 0,\ z = y^3,\ (1 \le y \le 2)$.

[30 Marks]

(d) Apply Charpit's method or otherwise to find the complete and the singular solution of the following nonlinear first-order PDE

$$16p^2z^2 + 9q^2z^2 + 4z^2 = 4.$$

Here, $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$.

[20 Marks]

Q6. (a) (i) Find the Fourier coefficients corresponding to the function

$$f(x) = \begin{cases} 2x, & 0 \le x < 3, \\ 0, & -3 < x < 0. \end{cases} \qquad \text{Period} = 6.$$

(ii) Write the corresponding Fourier series.

(iii) State where the discontinuities of $f(x)$ are located and to what value the series converges at these discontinuities.

[40 Marks]

(b) Use the finite Fourier transform to solve the following one-dimensional heat equation

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0, \quad 0 < x < 4, \quad t > 0,$$

subject to the boundary and initial conditions

$$U(0,t) = 0, \; U(4,t) = 0, \; U(x,0) = 2x.$$

[40 Marks]

(c) (i) Define the *gamma-function* $\Gamma(x)$ and *beta-function* $B(m,n)$, where m, n are positive integers.

(ii) Evaluate the integral

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}}.$$

(You may use the following results without proof

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.)$$

[20 Marks]

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