EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE - 2013/2014

SECOND SEMESTER (April/May, 2016)

PM 102 - REAL ANALYSIS

(Proper & Repeat)

...all Questions Time: Three hours

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a) i. Define what is meant by an inductive set in the set of real numbers, $\mathbb{R}$.

[10 Marks]

ii. Prove that the set of natural numbers, $\mathbb{N}$ is the smallest inductive set in $\mathbb{R}$.

[20 Marks]

b) i. Define the terms '*Supremum*' and '*Infimum*' of a non-empty subset of $\mathbb{R}$.

[10 Marks]

ii. State the completeness property of $\mathbb{R}$, and use it to prove that every non-empty bounded below subset of $\mathbb{R}$ has an infimum.

[30 Marks]

c) Consider the set $T = \left\{(-1)^n\left(1 - \frac{1}{n}\right) \ : \ n \in \mathbb{N}\right\}$.

i. Prove that 1 is an upper bound of T. [15 Marks]

ii. Prove that if d is an upper bound of T, then $1 \leqslant d$. [15 Marks]

iii. Use (a) and (b) to prove that $Sup\ T = 1$. [10 Marks]

Q2. (a) State what it means by a sequence of real numbers (x_n) converges to a limit

[10 Marks]

Use the definition to show that

$$\lim_{n\to\infty}\left(\frac{3n-1}{4n+5}\right)=\frac{3}{4}.$$

[20Marks]

(b) Prove that every convergent sequence of real numbers is bounded. [20 Marks]

(c) State the Monotone Convergent Theorem. [10 Marks]

Let $x_1=\sqrt{2}$ and $x_{n+1}=\sqrt{2+x_n}$ for $n=1,2,3,\ldots.$

i. Show that(x_n) is an monotonically increasing sequence. [15 Marks]

ii. Show that $x_n\leq 2$ for all $n\in\mathbb{N}$. [10 Marks]

iii. Does the sequence converge at all? Justify your answer. [15 Marks]

Q3. (a) Define the following terms:

i. a *subsequence* of a sequence; [10 Marks]

ii. *Cauchy* sequence. [10 Marks]

(b) Use the result, a real sequence (x_n) converges to a real number l, then any subsequence of (x_n) converges to the same limit l, to prove that

$$\lim_{n\to\infty} c^{\frac{1}{n}}=1 \text{ for } c>1.$$

[30 Marks]

(c) Let (a_n) and (b_n) be two Cauchy sequences and $c_n=|a_n-b_n|$. Show that (c_n) is a Cauchy sequence.

[20 Marks]

(d) Prove that a sequence (x_n) of real numbers is convergent if and only if it is a Cauchy sequence .

[30 Marks]

Q4. (a) Let $f:\mathbb{R}\to\mathbb{R}$ be a function. Explain what is meant by the function f has a limit $l(\in\mathbb{R})$ at a point $a(\in\mathbb{R})$.

Prove that $\lim_{x\to 5} x^2-3x+1=11$.

[25 Marks]

(b) Prove that limits of a function, when they exist, are unique. [15 Marks]

(c) Let $A \subseteq \mathbb{R}$ and $f : A \longrightarrow \mathbb{R}$ be a function. Prove that $\lim_{x \longrightarrow a} f(x) = l$ if and only if for every sequence (x_n) in A with $x_n \longrightarrow a$ as $n \longrightarrow \infty$ such that $x_n \neq a$ for all $n \in \mathbb{N}$, we have $f(x_n) \longrightarrow l$ as $n \longrightarrow \infty$.

[30 Marks]

(d) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function and $\lim_{x \longrightarrow a} f(x) = l(\neq 0)$.

Prove the following:

i. there exist $\delta > 0$ such that $\frac{|l|}{2} < |f(x)| < \frac{3|l|}{2}$, for all x such that $0 < |x - a| < \delta$; [15 Marks]

ii. $\lim_{x \longrightarrow a} \frac{1}{f(x)} = \frac{1}{l}$, if $f(x) \neq 0$, $\forall x \in \mathbb{R}$. [15 Marks]

(a) Define what it means to say that a real-valued function f is continuous at a point $'a'$ in its domain.

Let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if} \quad x \neq 0; \\ 1 \ \text{if} & \quad x = 0. \end{cases}$$

Prove that, f is continuous at $x = 0$. [25 Marks]

(b) Show that if $\lim_{x \to a} f(x) = l$, then $\lim_{x \to a} |f(x)| = |l|$. Is the converse of this result true? Justify your answer. [25 Marks]

(c) Prove that if a function $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$, then it is bounded on $[a, b]$. [30 Marks]

(d) State the Intermediate Value Theorem and use it to show that the equation $2x^2(x + 2) - 1 = 0$ has a root in each of the intervals $(-2, -1)$, $(-1, 0)$ and $(0, 1)$. [20 Marks]

(a) State what is meant by the statement that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is

i. differential at $a(\in \mathbb{R})$, [10 Marks]

ii. strictly decreasing at $a(\in \mathbb{R})$. [10 Marks]

(b) Prove that if a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable at $a (\in \mathbb{R})$ and $f'(a) < 0$

then f is strictly decreasing at a.

Is the converse true? Justify your answer.

[30 Marks]

(c) State the *Rolle's Theorem*, and use it to prove the *Mean Value Theorem*.

If the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b) and

$f'(x) = 0, \forall x \in [a, b]$, prove that f is a constant function on $[a, b]$.

[50 Marks]