EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE - 2008/2009

SECOND SEMESTER (Jun./Sept, 2015)

EXTERNAL DEGREE

EXTMT 103 - VECTOR ALGEBRA & CLASSICAL MECHANICS I

(REPEAT)

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er all questions | Time : Three hours

(a) For any three vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, prove that

$$\underline{a} \wedge (\underline{b} \wedge \underline{c}) = (\underline{a} \cdot \underline{c})\ \underline{b} - (\underline{a} \cdot \underline{b})\ \underline{c}.$$

(b) Let the vector $\underline{x}$ be given by the equation $\lambda \underline{x} + \underline{x} \wedge \underline{a} = \underline{b}$, where $\underline{a}, \underline{b}$ are constant vectors and λ is a non-zero scalar. Show that $\underline{x}$ satisfies the equation

$$\lambda^2(\underline{x} \wedge \underline{a}) + (\underline{a} \cdot \underline{b})\underline{a} - \lambda|\underline{a}|^2\underline{x} + \lambda(\underline{a} \wedge \underline{b}) = 0.$$

Hence find $\underline{x}$ in terms of $\underline{a}, \underline{b}$ and λ.

(c) Find the vector $\underline{x}$ and the scalar λ which satisfy the equations

$$\underline{a} \wedge \underline{x} = \underline{b} + \lambda\underline{a}, \qquad \underline{a} \cdot \underline{x} = 2,$$

where $\underline{a} = \underline{i} + 2\underline{j} - \underline{k}$ and $\underline{b} = 2\underline{i} - \underline{j} + \underline{k}$.

2. (a) Define the following terms:

   i. the gradient of a scalar field ϕ;

   ii. the divergence of a vector field $\underline{A}$;

   iii. the curl of a vector field $\underline{A}$.

   Prove that if ϕ is a scalar field and A is a vector field then

$$\text{curl}(\phi \underline{A}) = \phi \ \text{curl}\underline{A} + \text{grad}\phi \wedge \underline{A}.$$

   (b) Let $\underline{r} = x\ \underline{i} + y\ \underline{j} + z\ \underline{k}$ and $r = |\underline{r}|$. If $\underline{a}$ is a constant vector, find

   - $\text{grad}(\underline{a} \cdot \underline{r})$;
   - $\text{curl}(\underline{a} \wedge \underline{r})$.

   Hence show that

   i. $\text{grad}\left(\dfrac{\underline{a} \cdot \underline{r}}{r^3}\right) = \dfrac{\underline{a}}{r^3} - 3\dfrac{(\underline{a} \cdot \underline{r})}{r^5}\ \underline{r}$;

   ii. $\text{curl}\dfrac{(\underline{a} \wedge \underline{r})}{r^3} = \dfrac{2\underline{a}}{r^3} + 3\dfrac{(\underline{a} \wedge \underline{r})}{r^5} \wedge \underline{r}$.

3. State the Stokes' theorem.

   (a) Verify the Stokes' theorem for a vector $\underline{A} = (2x - y)\ \underline{i} - yz^2\ \underline{j} - y^2 z\ \underline{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

   (b) Evaluate $\iiint_V \phi\ dV$, where $\phi = 45x^2 y$ and V is the closed region bounded by the planes $4x + 2y + z = 8,\ x = 0,\ y = 0,\ z = 0.$.

4. A particle of mass m rests on a smooth horizontal table attached through a fixed point on the table by a light elastic string of modules mg and unstretched length $'a'$. Initially the string is just taut and the particle is projected along the table in a direction perpendicular to the line of the string with velocity $\sqrt{\dfrac{4ag}{3}}$. Prove that if r is the distance of the particle from the fixed point at time t then

$$\frac{d^2\underline{r}}{dt^2} = \frac{4ga^3}{3r^3} - \frac{g(r-a)}{a}.$$

Prove also that the string will extend until its length is $2a$ and that the velocity of the particle is then half of its initial velocity.

particle moves in a plane with the velocity v and the tangent to the path of the rticle makes an angle ψ with a fixed line in the plane. Prove that the components of celeration of the particle along the tangent and perpendicular to it are $\dfrac{dv}{dt}$ and $v\dfrac{d\psi}{dt}$ spectively.

body attached to a parachute is released from an aeroplane which is moving hor-ontally with velocity v_0. The parachute exerts a drag opposing motion which is k mes the weight of the body , where k is a constant. Neglecting the air resistance to e motion of the body, prove that if v is the velocity of the body when its path is clined an angle ψ to the horizontal, then

$$v = \frac{v_0 \sec\psi}{(\sec\psi + \tan\psi)^k} \ .$$

rove that if $k = 1$, the body cannot have a vertical component of velocity greater han $\dfrac{v_0}{2}$.

tate the angular momentum principle for motion of a particle.

right circular cone with a semi vertical angle α is fixed with its axis vertical and ertex downwards. A particle of mass m is held at the point A on the smooth inner urface of the cone at a distance $'a'$ from the axis of revolution. The particle is rojected perpendicular to OA with velocity $'u'$, where O is the vertex of the cone. Show that the particle rises above the level of A if $u^2 > ag\cot\alpha$ and greatest reaction etween the particle and the surface is

$$mg\left(\sin\alpha + \frac{u^2}{ag}\cos\alpha\right).$$