62

EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST YEAR EXAMINATION IN SCIENCE - 2012/2013

SECOND SEMESTER - (AUG./SEPT., 2015)

AM 104 - DIFFERENTIAL EQUATIONS

AND

FOURIER SERIES

(PROPER & REPEAT)

Answer **All** Questions   Time Allowed: **3** Hours

Q1. (a) State the necessary and sufficient condition for the ordinary differential equation (ODE)

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

to be *exact*. **[10 Marks]**

Find the general solution of the following ODE

$$(x\sqrt{x^2+y^2} - y) + (y\sqrt{x^2+y^2} - x)\frac{dy}{dx} = 0.$$

[40 Marks]

(b) Find the general solution of the following ODE multiplying it by an appropriate integrating factor

$$(2xy + 4x^3)dx + (x^2 + x^2y + x^4)dy = 0.$$

[30 Marks]

(c) Solve the following nonlinear first-order Bernoulli's equation

$$x\frac{dy}{dx} + y - x^3y^6 = 0.$$

[20 Marks]

Q2. Let $D \equiv d/dx$ be a differential operator. Show that a particular integral of the ODE

$$(D - \alpha)(D - \beta)y = P(x),$$

where α, β are arbitrary real constants and $P(x)$ is an arbitrary function in its variable, is given by

$$y = e^{\alpha x} \int e^{(\beta-\alpha)x} \left(\int Pe^{-\beta x}\, dx \right) dx.$$

[40 Marks]

Using the above result or otherwise, obtain the general solution of the following ODE:

(i) $(D^2 + D - 2)y = 2(1 + x - x^2)$;

(ii) $(D^2 - 9D + 18)y = e^{e^{-3x}}$.

[60 Marks]

Q3. (a) Let $x = e^t$. Show that

$$x\frac{d}{dx} \equiv \mathcal{D}, \quad x^2\frac{d^2}{dx^2} \equiv \mathcal{D}^2 - \mathcal{D},$$

and

$$x^3\frac{d^3}{dx^3} \equiv \mathcal{D}(\mathcal{D} - 1)(\mathcal{D} - 2),$$

where $\mathcal{D} \equiv \dfrac{d}{dt}$. [20 Marks]

Use the above results to find the general solution of the following Cauchy-Euler differential equation

$$(x^3D^3 + 2xD - 2)y = x^2 \ln x + 3x,$$

where $D \equiv \dfrac{d}{dx}$. [40 Marks]

(b) Define what is meant by *orthogonal trajectories* of curves.

[10 Marks]

Find the orthogonal trajectories of the family of curves

$$r = a(1 + \sin\theta),$$

where a is a constant. [30 Marks]

Q4. (a) Define what is meant by the point, $x = x_0$, being

(i) an *ordinary* ;

(ii) a *singular*;

(iii) a *regular singular*

64

point of the DE

$$y'' + p(x)y' + q(x)y = 0,$$

where the prime denotes differentiation with respect to x, and $p(x)$ and $q(x)$ are rational functions.

[30 Marks]

(b) (i) Find the regular singular point(s) of the DE

$$xy'' + (x+1)y' + 2y = 0. \qquad (1)$$

(ii) Use the method of Frobenius to find the general solution of the equation (1).

[70 Marks]

Q5. (a) Solve the following system of DEs:

(i) $\dfrac{dx}{1} = \dfrac{dy}{1} = \dfrac{dz}{(x+y)(1+2xy+3x^2y^2)}$;

(ii) $\dfrac{dx}{x} = \dfrac{dy}{x+z} = \dfrac{dz}{-z}$.

[30 Marks]

(b) Write down the condition of integrability of the total differential equation

$$P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz = 0.$$

[5 Marks]

Hence solve the following equation

$$x\,dx + y\,dy + (x^2+y^2+z^2+1)z\,dz = 0.$$

[15 Marks]

(c) Find the equation of the integral surface satisfying the linear partial differential equation (PDE),

$$x(y-z)p + y(z-x)q = z(x-y),$$

and passing through the curve $x = y = z$.

[30 Marks]

(d) Apply Charpit's method or otherwise to find the complete and the singular solution of the following nonlinear first-order PDE

$$p = (qy+z)^2.$$

Here, $p = \dfrac{\partial z}{\partial x}$ and $q = \dfrac{\partial z}{\partial y}$.

[20 Marks]

65

Q6. (a) Find the Fourier series of the function $f(x)$ given by

$$f(x) = e^x, \quad -\pi < x < \pi.$$

Hence derive a series for $\pi/\sinh\pi$.

[40 Mar

(b) Use the finite Fourier transform to solve the following one-dimensi heat equation

$$\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} = 0, \quad 0 < x < \pi, \quad t > 0,$$

subject to the boundary and initial conditions

$$U(0,t) = 0, \; U(\pi,t) = 0, \; U(x,0) = 1.$$

[40 Marks

(c) (i) Define the *gamma-function* $\Gamma(x)$ and *beta-function* $B(m,n)$ where m, n are positive integers.

(ii) Evaluate the integral

$$\int_0^1 x^4(1-x)^2\,dx.$$

(You may use the following results without proof

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.)$$

[20 Marks

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