

EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
FIRST YEAR EXAMINATION IN SCIENCE - 2009/2010
SECOND SEMESTER - (APRIL/MAY, 2012)
AM 104 - DIFFERENTIAL EQUATIONS

AND
FOURIER SERIES
(PROPER)



Answer All Questions

Time Allowed: 3 Hours

- Q1. (a) State the necessary and sufficient condition for the differential equation (DE)

$$M(x, y) dx + N(x, y) dy = 0$$

to be exact.

[10 Marks]

Prove that $x^a y^b$ is an integrating factor of the DE

$$kydx + lxdy + x^c y^d (mydx + nxdy) = 0$$

if

$$\frac{a+1}{k} = \frac{b+1}{l} \quad \text{and} \quad \frac{a+c+1}{m} = \frac{b+d+1}{n},$$

where $a, b, c, d, k(\neq 0), l(\neq 0), m(\neq 0)$ and $n(\neq 0)$ are constants.

Hence integrate the DE

$$3ydx - 2xdy + x^2 y^{-1} (10ydx - 6xdy) = 0.$$

[50 Marks]

- (b) Show that there are two values of the constant α for which α/x is a solution of the Riccati's equation

$$\frac{dy}{dx} - \frac{2}{x^2} + y^2 = 0, \quad (1)$$

and hence obtain the general solution of the equation (1).

[40 Marks]

Q2. (a) If $F(D) = \sum_{i=0}^n p_i D^i$, where $D \equiv d/dx$ is a differential operator and $p_i, i = 1, \dots, n$, are constants with $p_0 \neq 0$, then prove the following formulas:

(i) $\frac{1}{F(D)} e^{\alpha x} = \frac{1}{F(\alpha)} e^{\alpha x}$, α is a constant and $F(\alpha) \neq 0$;

(ii) $\frac{1}{F(D)} e^{\alpha x} V = e^{\alpha x} \frac{1}{F(D + \alpha)} V$, where V is a function only of x .

[30 Marks]

(b) Find the general solution of the following DEs by using the results in (a):

(i) $(D - 1)y = (x + 3) e^{2x}$;

(ii) $(D^2 - 5D + 6)y = x^3 e^{2x}$.

[30 Marks]

(c) Show that

$$\frac{\cos \beta x - \cos(\beta + h)x}{(\beta + h)^2 - \beta^2}$$

is a solution of the DE

$$(D^2 + \beta^2)y = \cos(\beta + h)x,$$

where β and h are constants.

Hence deduce the particular integral of the DE

$$(D^2 + \beta^2)y = \cos \beta x.$$

[40 Marks]

Q3. (a) Prove, by induction or otherwise, that if $T \equiv x(d/dx) = xD$, where $D \equiv d/dx$, then

$$x^n \frac{d^n y}{dx^n} = T(T - 1)(T - 2) \dots (T - n + 1)y,$$

where n is a positive integer.

Hence or otherwise show that the solution of the DE

$$x^2 \frac{d^2 y}{dx^2} - 4x \frac{dy}{dx} + 6y = x^5$$

is

$$y(x) = Ax^2 + Bx^3 + \frac{1}{6}x^5,$$

where A and B are arbitrary constants.

[70 Marks]

(You may use the following identities without proof:

(i) $F(T)x^m = x^m F(m)$;

(ii) $\frac{1}{F(T)}x^m = \frac{x^m}{F(m)}$; provided $F(m) \neq 0$, where m is some constant.)

(b) With $D \equiv d/dx$, solve the following system of DEs

$$\begin{aligned}(5D + 4)y - (2D + 1)z &= e^{-x}, \\ (D + 8)y - 3z &= 5e^{-x}.\end{aligned}$$

[30 Marks]

Q4. (a) Define what is meant by the point, ' $x = x_0$ ', being

- (i) an *ordinary* ;
- (ii) a *singular*;
- (iii) a *regular singular*

point of the DE

$$y'' + p(x)y' + q(x)y = 0,$$

where the prime denotes differentiation with respect to x , and $p(x)$ and $q(x)$ are rational functions.

[30 Marks]

(b) (i) Find the regular singular point(s) of the DE

$$9x^2y'' + 9x^2y' + 2y = 0. \quad (2)$$

(ii) Use the method of Frobenius to find the general solution of the equation (2).

[70 Marks]

Q5. (a) Solve the following system of DEs:

(i) $\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y - 3x)}$;

(ii) $\frac{dx}{y + z} = \frac{dy}{z + x} = \frac{dz}{x + y}$.

[30 Marks]

(b) Write down the condition of integrability of the total differential equation

$$P(x, y, z) dx + Q(x, y, z) dy + R(x, y, z) dz = 0.$$

[5 Marks]

Hence solve the following equation

$$(y^2 + z^2 - x^2) dx - 2xy dy - 2xz dz = 0. \quad (3)$$

(You may use the integrating factor $\mu = 1/x^2$ for equation (3).)

[15 Marks]

- (c) Find the general solution of the following linear first-order partial differential equations:

(i) $xp + yq = z;$

(ii) $yp - xq = y^2 - x^2.$

[30 Marks]

- (d) Apply Charpit's method or otherwise to find the complete and the singular solution of the following non-linear first-order partial differential equation

$$p^2 + yq + 2z + 2y^2 = 0.$$

Here, $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}.$

[20 Marks]

- Q6. (a) Show that the Fourier series of the function

$$f(x) = x^2, \quad -\pi \leq x \leq \pi$$

can be expressed as

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos n\pi}{n^2}.$$

Hence deduce that

[20 Marks]

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

[20 Marks]

- (b) Use Fourier transform to solve the one-dimensional heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \quad x > 0, \quad t > 0$$

subject to the conditions

$$u(0, t) = 0, \quad u(x, 0) = \begin{cases} 1, & 0 < x < 1, \\ 0, & x > 1 \end{cases}$$

and $u(x, t)$ is bounded.

[40 Marks]

- (c) (i) Define the *gamma-function* $\Gamma(x)$ and *beta-function* $B(m, n)$, where m, n are positive integers.

(ii) Evaluate the integral

$$\int_0^1 \frac{1}{\sqrt{1-x^4}} dx.$$

(You may use the following results without proof

$$B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \quad \text{and} \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.)$$

[20 Marks]
