EASTERN UNIVERSITY, SRI LANKA

SECOND YEAR EXAMINATION IN SCIENCE - 2012/2013

FIRST SEMESTER(April/May, 2015)

PM 203 - EIGENSPACE AND QUADRATIC FORMS

Answer all Questions

Time: Two hours

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1. (a) Define the terms *eigenvalue* and *eigenvector* of a linear transformation.

   Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$

   (b) i. Prove that the eigenvalues of a Hermitian matrix are real.

   ii. Show that a matrix A and its transpose A^T have the same characteristic polynomial.

   iii. Let A and B be n-square matrices. Show that AB and BA have the same eigenvalues.

   (c) Orthogonally diagonalize the matrix

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

2. Define the term *minimum polynomial* of a square matrix.

(a) State the *Cayley-Hamilton* theorem.

Find the minimum polynomial of the square matrix

$$\begin{pmatrix} 2 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}.$$

(b) Prove the following:

(i) If $m(t)$ is the minimum polynomial of a matrix A and $f(t)$ is a polynomi[al] such that $f(A) = 0$, then $m(t)$ divides $f(t)$.

(ii) The characteristic and minimum polynomials of a matrix A have the sa[me] irreducible factors.

(c) Let $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where A, B are square matrices and 0 denotes the ze[ro] matrix. Show that the minimum polynomial $m(t)$ of M is the least comm[on] multiple of the minimum polynomials $g(t)$ and $h(t)$ of A and B, respectiv[ely].

3. (a) Find an orthogonal transformation which reduces the following quadratic fo[rm] to a diagonal form

$$5x_1^2 + 6x_2^2 + 7x_3^2 - 4x_1x_2 + 4x_2x_3 = 1.$$

(b) Simultaneously diagonalize the following pair of quadratic forms

$$\phi_1 = x_1^2 + 2x_2^2 + 8x_2x_3 + 12x_1x_2 + 12x_1x_3,$$

$$\phi_2 = 3x_1^2 + 2x_2^2 + 5x_3^2 + 2x_2x_3 - 2x_1x_3.$$

4. (a) Define what is meant by an inner product on a vector space.

Let $x = (x_1, x_2, ..., x_n)$, $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$, where $x_i, y_i \in \mathbb{R}$, $i = 1, 2,$

Let the inner product $< .,. >$ be defined on $\mathbb{R}^n$ as

$$< x, y >= xy^T = \sum_{i=1}^{n} x_i y_i.$$

Show that $(\mathbb{R}^n, < .,. >)$ is an inner product space.

(b) Prove that if M is a subspace of a finite inner product space V, then $V = M \oplus M^{\perp}$, where $M^{\perp}$ is the orthogonal complement of M.

(c) State the Gram Schmidt process.

Find the orthonormal set for span of M in $\mathbb{R}^4$, where

$$M = \{(1,1,1,1)^T, (1,1,2,4)^T, (1,2,-4,-3)^T\}.$$