EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SECOND EXAMINATION IN SCIENCE - 2011/2012

FIRST SEMESTER (Apr./May, 2014)

AM 207 - NUMERICAL ANALYSIS

(Proper & Repeat)

nswer all questions — Time : Two hours

1. (a) Define what is meant by:

   i. *absolute error*;

   ii. *relative error* .

   Let $p = 0.54617$ and $q = 0.54601$. Use four-digit arithmetic to approximate $p - q$, and determine the absolute and relative errors when rounding and chopping.

   (b) Let $f(x) = \dfrac{1}{1+x}$.

   i. Compute the n^{th} derivative of $f(x)$ and the Taylor polynomial $P_n(x)$ of $f(x)$ for $x_0 = 0$.

   ii. Let $R_n(x)$ denotes the remainder term $f(x) - P_n(x)$ in Taylor's Theorem. For each n, show that $|R_n(x)| = |x|^{n+1}$ for all $x \geq 0$.

   iii. Show that

   $$\arctan x = \int_0^x \frac{1}{1+t^2}\, dt = \int_0^x f(t^2)dt.$$

   iv. For $n = 1,\ 2$, approximate $\arctan\left(\dfrac{1}{2}\right)$ by using $\displaystyle\int_0^x P_n(t^2)dt$.

2. (a) Let $x = \phi(x)$ be the rearrangement of the equation $f(x) = 0$ and define the iteration,

$$x_{n+1} = \phi(x_n), \qquad n = 0, 1, ...$$

with the initial value x_0. If $\phi'(x)$ exists and is continuous such that $|\phi'(x)| \leq K < 1$ for all x, where K is a positive constant, then show that the sequence (x_n) generated by the above iteration converges to the unique root α of the equation $f(x) = 0$.

The equation $x \cos x = x \sin x$ has a root at $x = \pi/4$. Which of the iteration processes: $x_{i+1} = x_i \tan x_i$ or $x_{i+1} = x_i \cot x_i$ should be used to find this root?

(b) Obtain Newton Raphson method to compute the root of the equation $f(x) = 0$ in an interval $[a, b]$.

Use the Newton-Rapshon method to find a positive real root of $\cos x - x^3 =$ correct to four decimal places.

3. (a) If $f \in C^{n+1}[a, b]$ and P_n is the Lagrange's interpolating polynomial which interpolates the function $f(x)$ at the distinct points $x_0, x_1, ..., x_n$ in $[a, b]$, prove that for all $x \in [a, b]$, there exists $\xi \in (a, b)$ such that the error, $E(x)$, in the interpolation is given by

$$E(x) = \frac{\prod_{n+1}(x)}{(n+1)!} f^{n+1}(\xi),$$

where $\prod_{n+1}(x) = (x - x_0)(x - x_1)...(x - x_n)$.

(b) Let $f(x) = \sqrt{x}$.

i. Compute the second degree interpolating polynomial, $P_2(x)$, for $f(x)$ using the points $x_0 = 1$, $x_1 = 2.25$ and $x_2 = 4$.

ii. Evaluate $P_2(2)$ and use the interpolation error theorem to estimate the error in this approximation of $\sqrt{2}$.

iii. Compute the actual error and compare with the estimated value in part (ii).

4. (a) With the usual notations, the Simpson's rule is given by

$$\int_{x_{i-1}}^{x_{i+1}} f(x)dx = \frac{h}{3}\left(f_{i-1} + 4f_i + f_{i+1}\right) - \frac{1}{90}h^5 f^{(iv)}(\xi_i), \text{ where } \xi_i \in [x_{i-1}, x_{i+1}].$$

Obtain the composite Simpson's rule, and show that the composite error is less than or equal to

$$\frac{1}{180}h^4(b-a)\left|f^{(iv)}(\xi)\right|, \text{ where } \left|f^{(iv)}(\xi)\right| = \max_{a\le x\le b}\left|f^{(iv)}(x)\right|.$$

Hence show that composite Simpson's rule is exact for all polynomials of degree 3 or less.

(b) Find the solution of the system of equations

$$\begin{aligned} 45x_1 &+ 2x_2 &+ 3x_3 &= 58 \\ -3x_1 &+ 22x_2 &+ 2x_3 &= 47 \\ 5x_1 &+ x_2 &+ 20x_3 &= 67, \end{aligned}$$

correct to three decimal places, using the Gauss-Seidel iteration method.