EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

SECOND EXAMINATION IN SCIENCE - 2012/2013

FIRST SEMESTER (April/May, 2015)

PM 201 - VECTOR SPACES AND MATRICES

Answer all questions | Time: Three hours

1. (a) i. Let V be a vector space over a field $\mathbb{F}$. Prove that, a subset S of V is a subspace of V if the following conditions are satisfied:

a. $S \neq \Phi$;

b. $x + y \in S$ for any $x, y \in S$;

c. $\alpha x \in S$ for any $\alpha \in \mathbb{F}$ and $x \in S$.

ii. Which of the following sets are subspaces of $\mathbb{R}^2$ under usual addition and scalar multiplication of vectors? Justify your answer.

a. $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in \mathbb{Z} \right\}$;

b. $\left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 x_2 = 0, x_1, x_2 \in \mathbb{R} \right\}$.

(b) Prove that, $V = \{f \in C[a,b] : f((a+b)/2) = 0\}$, is a vector space with usual addition of functions and scalar multiplication, where the set $C[a,b]$ denotes the set of all real-valued continuous functions defined on the interval $[a,b] \subseteq \mathbb{R}$.

2. (a) Define what is meant by

i. a *linearly independent set* of vectors;

ii. a *basis* of a vector space.

(b) Let V be a vector space. Show that

i. any linearly independent set of vectors of V may be extended to a basis fo V;

ii. if L is a subspace of V, then there exists a subspace M of V such tha $V = L \oplus M$, where $\oplus$ denotes the direct sum.

(c) Let V be a vector space over the field $\mathbb{F}$.

i. Let $\{u_1, u_2, \cdots, u_n\}$ be a linearly dependent subset of V with each $u_j \neq 0,\ j = 1, 2, \cdots, n$. Prove that, there exist $u_i,\ 2 \leq i \leq n$, which is a linear combination of the preceding vectors.

ii. Let S be a subset of V and $u, v \in V$. If $u \in \langle S \cup \{v\} \rangle$ and $u \notin \langle S \rangle$, then prove that $v \in \langle S \cup \{u\} \rangle$.

iii. Let S, T be two subspaces of V with $S \cap T = \{0\}$ and let $\{s_1, s_2, \cdots, s_m\}$ and $\{t_1, t_2, \cdots, t_n\}$ be linearly independent subsets of S and T, respectively. Prove that $\{s_1, s_2, \cdots, s_m, t_1, t_2, \cdots, t_n\}$ is a linearly independent subset of V.

3. (a) Define what is meant by

i. the *range space* $R(T)$;

ii. the *null space* $N(T)$

of a linear transformation T from a vector space V into another vector space W. Find $R(T)$ and $N(T)$ of the linear transformation $T : \mathbb{P}_2 \to \mathbb{R}^2$ defined by

$$T(\alpha + \beta x + \gamma x^2) = (\alpha - \beta, \beta - \gamma),$$

where $\mathbb{P}_2$ is the set of polynomials in one variable with real coefficients and of degree less than or equal to 2.

Verify the equation, $\dim \mathbb{P}_2 = \dim(R(T)) + \dim(N(T))$, for the above linear transformation T.

(b) Let the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x, y, z) = (x + 2y,\ x + y + z,\ z)$$

and let $B_1 = \{(1, 1, 1),\ (1, 2, 3),\ (2, -1, 1)\}$ and $B_2 = \{(1, 1, 0),\ (0, 1, 1), (1, 0, 1)\}$ be bases of $\mathbb{R}^3$.

i. find the matrix representation of T with respect to the basis B_1;

ii. using the transition matrix, find the matrix representation of T with respect to the basis B_2.

4. (a) Define what is meant by

(i) *elementary matrix*;

(ii) *row reduced echelon form* of a matrix.

(b) Let A be a non-singular matrix. Prove the following:

(i) the inverse matrix of A, A^{-1}, can be obtained by applying the same finite elementary row operations, when A is transformed into the identity matrix;

(ii) if B is a matrix obtained by performing an elementary row operation on A, then A and B have the same rank.

(c) Find the rank of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 3 & 3 & 0 & 2 \\ 2 & 1 & 3 & 3 & -1 & 3 \\ 2 & 1 & 1 & 1 & -2 & 4 \end{pmatrix}.$$

(d) Find the row reduced echelon form of the matrix

$$\begin{pmatrix} 1 & 3 & -1 & 2 \\ 0 & 11 & -5 & 3 \\ 2 & -5 & 3 & 1 \\ 4 & 1 & 1 & 5 \end{pmatrix}.$$

5. Define what is meant by the matrix, *adjoint of A*, as applied to an $n \times n$ matrix $A = (a_{ij})$.

(a) With the usual notations, prove that

$$A \cdot (adj A) = (adj A) \cdot A = det A \cdot I.$$

Hence prove that, $adj(adj\ A) = (det\ A)^{n-2}A$, where $n \in \mathbb{N}$.

(b) Prove that, if $A = \begin{pmatrix} 2a & -a^2 \\ 1 & 0 \end{pmatrix}$, then $A^n = \begin{pmatrix} (n+1)a^n & -na^{n+1} \\ na^{n-1} & (1-n)a^n \end{pmatrix}$.

(c) By applying the appropriate row or column operations, prove that the determina
of the matrix

$$\begin{pmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{pmatrix}$$

can be expressed as

$$abcd\,(1 + 1/a + 1/b + 1/c + 1/d),$$

where $a, b, c, d \in \mathbb{R} \setminus \{0\}$.

6. (a) State the necessary and sufficient condition for a system of linear equations to be consistent.

Let the following system of linear equations be given

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_1 + 2x_2 + 3x_3 &= 10 \\ x_1 + 2x_2 + \lambda x_3 &= \mu. \end{aligned}$$

Investigate for what values of λ, μ, the above system of equations have

i. a unique solution;

ii. an infinite number of solutions;

iii. no solution.

(b) State and prove Crammer's rule for 3 × 3 matrix, and use it to solve the following system of linear equations

$$\begin{aligned} 5x_1 - x_2 + 3x_3 &= 10 \\ 6x_1 + 4x_2 - x_3 &= 19 \\ x_1 - 7x_2 + 4x_3 &= -15. \end{aligned}$$