EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
SECOND EXAMINATION IN SCIENCE - 2012/2013
SECOND SEMESTER (Oct./Nov., 2015)
PM 202 - METRIC SPACE
(Proper & Repeat)

Answer all questions **Time: Two Hours**

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1. Define what is meant by a

- *metric space.*
- *complete metric space.*

(a) Let X be the set of all bounded sequences of real numbers. Define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i}$$

where $x = \{x_i\}$ and $y = \{y_i\}$ are two arbitrary elements of X. Show that (X, d) is a metric space.

(b) Define what is meant by a *Cauchy* sequence in a metric space.

Let X be the set of all positive integers. Define $d : X \times X \to \mathbb{R}$ by

$$d(m, n) = \left|\frac{1}{m} - \frac{1}{n}\right|.$$

Show that (X, d) is a metric space but not complete.

Find the distance between the points $(1, 1, 1, ...)$ and $(2, 2, 2, ...)$.

2. (a) Let (X, d) be a metric space and let A be a subset of X. Define what is meant by

- *interior* A° *of* A.
- *closure* $\overline{A}$ *of* A.

i. Prove that A° is the largest open set contained in A and that $\overline{A}$ is the smallest closed set containing A.

ii. Is it true that, arbitrary union of closed sets is closed? Justify your answer

(b) Define the following terms as applied to subsets of a metric space:

- *separated*;
- *disconnected.*

i. Prove that two open subsets of a metric space are separated if and only i they are disjoint.

ii. Prove that a metric space (X, d) is disconnected if and only if there exists nonempty proper subset of X which is both open and closed.

3. Define the term *compact* as applied to subsets of a metric space.

(a) Show that $[a, b]$ is a compact subset of $\mathbb{R}$ with respect to the usual metric.

(b) Let A be a compact subset of a metric space (X, d) and let $a \in X \setminus A$. Prove th there exist open sets G and H such that $a \in G$, $A \subseteq H$ and $G \cap H = \phi$. Henc show that any compact subset of X is closed.

4. Define what is meant by a *continuous function* between two metric spaces.

(a) Let (X, d_1) and (Y, d_2) be any two metric spaces, and $f : X \to Y$ be a functio Prove that f is continuous at $a \in X$ if and only if for every sequence $\{a_n\}$ in converging to a implies that $\{f(a_n)\}$ converges to $f(a)$.

(b) Let (X, d_1) and (Y, d_2) be any two metric spaces, and $f : X \to Y$ be a functi Prove that f is continuous if and only if $f^{-1}(G)$ is open in X whenever G is op in Y.

(c) Prove that $f : X \to Y$ is continuous if and only if $f^{-1}(B^\circ) \subseteq (f^{-1}(B))^\circ$.

(d) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2}, & \text{if} \quad (x,y) \neq (0,0); \\ 0, & \text{if} \quad (x,y) = (0,0) \end{cases}$$

where $\mathbb{R}^2$ and $\mathbb{R}$ are considered with respect to the usual metric. Discuss the continuity at the origin.