EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

THIRD EXAMINATION IN SCIENCE (2013/2014)

SECOND SEMESTER (June' 2016)

OCT 2017
UNIVERSITY, SRI LANKA

PM 301 - GROUP THEORY

Special Repeat

Answer all questions | Time: Three hours

---

1. (a) Define the following terms

   i. *group*;

   ii. *cyclic group*;

   Prove that every subgroup of a cyclic group is cyclic.

   Is the converse part true? Justify your answer.

   (b) State and prove *Lagrange's* theorem.

   i. In a group G, H and K are different subgroups of order p, p is prime. Show that $H \cap K = \{e\}$, where e is the identity element of G.

   ii. Prove that in a finite group G, the order of each element divides order of G. Hence prove that $x^{|G|} = e, \ \forall \, x \in G$.

2. (a) What is meant by saying that a subgroup of a group is *normal*?

   i. Let H and K be two normal subgroups of a group G. Prove that $H \cap K$ is a normal subgroup of G.

   ii. Prove that every subgroup of an abelian group G is a normal subgroup of G.

(b) With usual notations prove that

i. $N(H) \leq G$;

ii. $H \trianglelefteq N(H)$.

(c) Let $Z(G) = \{x \in G \mid xg = gx, \forall\, g \in G\}$. Prove the following

i. $Z(G) = \bigcap_{a \in G} C(a)$, where $C(a) = \{g \in G \ : \ ga = ag\}$

ii. $Z(G) \trianglelefteq G$.

3. (a) Define what is meant by two groups are *isomorphic*.

Let $G = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{R} \right\}$ be a group under the matrix multiplication. Prove that

i. the mapping $\phi : G \to (\mathbb{R}, +)$ defined by $\phi\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\right) = a$

is a homomorphism.

ii. G isomorphic to $\mathbb{R}$.

(b) State the *first isomorphism* theorem.

Let H and K be two normal subgroups of a group G such that $K \subseteq H$. Prove the following

i. $K \trianglelefteq H$;

ii. $H/K \trianglelefteq G/K$;

iii. $\dfrac{G/K}{H/K} \cong G/H$.

4. (a) Define *commutator subgroup* G' of a group G.

Prove that the following

i. $G' \trianglelefteq G$;

ii. G/G' is abelian.

(b) Let $H \trianglelefteq G$, $P = \{K \leq G \ : \ H \subseteq K\}$ and $Q = \{K' \ : \ K' \leq G/H$

Prove that there exists a one to one correspondence between P and Q.

5. (a) What is meant by the *internal direct product* as applied to a group. Is it true that all the groups satisfy the internal direct product property? Justify your answer.

   Let H and K be two subgroups of a group G, prove that G is a direct product of H and K if and only if

   i. each $x \in G$ can be uniquely expressed in the form $x = hk$, where $h \in H, k \in K$.

   ii. $hk = kh$ for any $h \in H, k \in K$.

   (b) Define the term $p-$*group*.

   Let G be a finite abelian group and let p be a prime number which divides the order of G. Prove that G has an element of order p.

6. (a) Define the following terms as applied to a permutation group.

   i. *cycle* of order r;

   ii. *transposition*;

   iii. *signature*.

   (b) Prove that every permutation in S_n can be expressed as a product of transpositions.

   (c) Prove that the set of all even permutations, A_n forms a normal subgroup of S_n and $|A_n| = \frac{n!}{2}$.

   (State any results you may use without proof)

   (d) i. find out wether the following permutation in S_8 is odd or even $\sigma = (1, 2, 8, 4)(4, 3, 2)(5, 7)(1, 4, 2, 3)$.

   ii. express σ as a product of disjoint cycles.