EASTERN UNIVERSITY, SRI LANKA

THIRD EXAMINATION IN SCIENCE - 2013/2014

FIRST SEMESTER (May/June, 2016)

PM 304 - GENERAL TOPOLOGY

Proper and Repeat

---

wer all questions | **Time: Two hours**

---

1. (a) Define what is meant by the term *topological space.*

   i. Let $f : X \to Y$ be a function from a non-empty set X into a topological space (Y, τ). Let $\sigma = \{f^{-1}(G) : G \in \tau\}$ be the class of inverses of open subsets of Y. Show that σ is a topology on X.

   ii. Let τ be the class consisting of the set of all real numbers $\mathbb{R}$, empty set ϕ and all open infinite intervals $A_n = (n, \infty)$ with $n \in \mathbb{Q}$ (the set of all rational numbers). Show that τ is not a topology on $\mathbb{R}$.

   (b) Suppose in part (ii) of (a), if $n \in \mathbb{R}$, find the *interior*, *exterior* and *boundary* of the closed infinite interval $A = [7, \infty)$.

   (c) Let (X, τ) be a topological space and $A \subseteq X$. Prove with the usual notations that $\overline{A} = A^o \cup b(A)$.

2. (a) Define what is meant by the statement that a function f from a topological space X into a topological space Y is continuous at a point $x \in X$.

   Prove the following:

   i. $f : X \to Y$ is continuous on X if and only if $f^{-1}(G)$ is open in X for each open set G in Y.

   ii. $f : X \to Y$ is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all subsets A of X.

   (b) i. Let $f : (X, \tau) \to (Y, \tau^*)$ and let S be a subbase for the topology τ^* on Y.

Prove that f is continuous if and only if the inverse of every member of the subbase S is an open subset of X.

ii. Let f be a function from a topological space X into the unit interval $[0,1]$. Use part (i) to show that if $f^{-1}[(a,1]]$ and $f^{-1}[[0,b)]$ are open subsets of X for all $0 < a, b < 1$ then f is continuous.

3. (a) Define what is meant by the term *connected set* in a topological space.

   (i) Let (X, τ) be a topological space. Prove that X is disconnected if and only if there are non-empty subsets A, B of X such that $X = A \cup B$ and $\overline{A} \cap B = \phi$ and $A \cap \overline{B} = \phi$.

   (ii) Let (X, τ_1) and (Y, τ_2) be two topological spaces and $f : X \to Y$ be a continuous function. Prove that the image of a connected subset A of X is connected in Y.

   (b) Define what is meant by the term *Hausdorff space.*

   Let τ be a topology on a real line $\mathbb{R}$ generated by the open-closed intervals $(a, b]$. Show that $(\mathbb{R}, \tau)$ is a Hausdorff space.

4. Prove or disprove the following statements:

   (a) continuous image of a compact set in a topological space is compact;

   (b) in the usual topology on $\mathbb{R}$, the set $(0, 1)$ is compact;

   (c) the class of open intervals $A_n = \left\{ \left(0, \frac{1}{n}\right) : n \in \mathbb{N} \right\}$ satisfies the finite intersection property and $\bigcap_{n \in \mathbb{N}} A_n = \phi$;

   (d) (X, τ) is a compact topological space if and only if for every class $\{F_i\}$ of closed subset of X, $\bigcap_i F_i = \phi$ implies $\{F_i\}$ contains a finite subclass $\{F_{i_1}, F_{i_2}, \cdots, F_{i_m}\}$ with $F_{i_1} \cap F_{i_2} \cap \cdots \cap F_{i_m} = \phi$.