EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE-1994/95 & 95/96

(Aug./Sep.97)

MT 203 & 204 - EIGEN SPACES & QUADRATIC FORMS AND DIFFERENTIAL GEOMETRY

Time Allowed: 02 Hours.

Answer only **four** questions selecting **two** from each section.

Section A

Eigen Spaces & Quadratic Forms

1. Define an eigenvalue and an eigenvector of a square matrix.

   (a) Let A be an $n \times n$ non-singular matrix and let $\Psi_A(t)$ denote the chracteristic polynomial of A. Show that

   $$\Psi_{A^{-1}}(t) = \frac{(-t)^n}{\det A}\Psi_A\left(\frac{1}{t}\right);\ (t \neq 0).$$

   Deduce that if $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues of A with algebraic multiplicities, then $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, ..., \frac{1}{\lambda_n}$ are eigenvalues of A^{-1} with algebraic multiplicities.

   (b) Prove that an $n \times n$ matrix A is similar to diagonal matrix D whose diagonal elements are eigenvalues of A if, and only if A has n linearly independent eigenvectors.

   Let $A = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{pmatrix}$. Find a non-sigular matrix P such that $P^{-1}AP$ is diagonal. Hence find the matrix A^n $(n \in \aleph)$.

2. (a) Define the terms "Symmetric" and "Skew-symmetric" as applied to an $n \times n$ matrix.
Prove that
   i. if A is an $n \times n$ real skew-symmetric matrix, then every eigenvalue of A is zero or purely imaginary;
   ii. if A is an $n \times n$ real matrix and λ is a eigenvalue of the real symmetric matrix $I_n + A^T A$ then $\lambda \geq 1$, where I_n is the $n \times n$ identity matrix;
   iii. if A is an $n \times n$ real symmetric matrix, then eigenvectors corresponding to distinct eigenvalues are orthogonal.

(b) Let $A = \begin{pmatrix} 5 & 2 & 4 \\ 2 & 8 & -2 \\ 4 & -2 & 5 \end{pmatrix}$. Find an orthogonal matrix P such that $P^T AP$ is diagonal.

3. (a) Reduce the quadratic form $3x_1^2 + 4x_2^2 + 2\sqrt{3}x_2x_3 + 6x_3^2$ to a diagonal form by using an orthogonal transformation.

(b) Let $X^T AX$ and $X^T BX$ be two quadratic forms of which $X^T BX$ is positive definite and let $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ be roots of the equation $\det(A - \lambda B) = 0$. Show that there exists a non-singular transformation $X = PY$ which reduces $X^T AX$ and $X^T BX$ to the form $\lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots\ldots + \lambda_n y_n^2$ and $y_1^2 + y_2^2 + \ldots\ldots + y_n^2$ respectively.

(c) Simultaneously reduce the following pair of quadratic forms.

$$\begin{aligned} Q_1 &= x_1^2 + 2x_2^2 + 8x_2x_3 + 12x_3x_1 + 12x_1x_3, \\ Q_2 &= 3x_1^2 + 2x_2^2 + 5x_3^2 + 2x_2x_3 - 2x_1x_3. \end{aligned}$$

Section B

Differential Geometry

4. State and prove Serret-Frenet formulae.

Let C be a curve of constant torsion τ and a point Q, a constant distance c from the point P on C, on the binormal to C at P. Show that the angle between the binormal to the locus of Q and the binormal to the given curve C is

$$\tan^{-1}\left[\frac{c\tau^2}{\kappa\sqrt{1+c^2\tau^2}}\right],$$

where κ is the curvature of the curve C at P.

5. Define the Involute and Evolute of a given curve C.

   (a) Find the vector equations of the Involute and Evolute of a given curve $C : \underline{r} = \underline{r}(s)$.

   (b) Find the involutes and evolutes of a curve $\underline{r} = (e^t, e^{-t}, \sqrt{2}t)$.

6. (a) Two curves C_1 and C_2 are called Bertrand curves if they have common principal normal lines. If the torsion $\tau \neq 0$ along the curve C, show that C is a Bertrand curve (i.e there exists a curve C_1 such that C and C_1 are Bertrand curves) if and only if, there are constants γ and α such that

$$\kappa + \gamma\tau = \frac{1}{\alpha},$$

   where κ is the curvature of the curve C.

   (b) Show that for a curve lying on a sphere of radius a and such that the torsion τ is never 0, the following equation is satisfied

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{\dot{\kappa}}{\kappa^2\tau}\right)^2 = a^2,$$

   where κ is the curvature of the curve.

   (c) Show that the necessary and sufficient condition for a curve to be a plane is $\tau = 0$.