EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE 1994/95 & 95/96

(August/September'97) – RE-REPEAT

MT205 & 208 – MATHEMATICAL METHODS & NUMERICAL ANALYSIS

Time: Two hours.

Answer only four questions selecting two from each section.

SECTION A
MATHEMATICAL METHODS

1. (a) i. Write the transformation equations for the following tensors.

$$A^i_{jk}, \quad B^{ijk}_l, \quad C_{mn}.$$

ii. If $\overline{A}^p_r = \dfrac{\partial \overline{x}^p}{\partial x^q}\dfrac{\partial x^s}{\partial \overline{x}^r} A^q_s$, then prove that $A^q_s = \dfrac{\partial x^q}{\partial \overline{x}^p}\dfrac{\partial \overline{x}^r}{\partial x^s}\overline{A}^p_r$.

(b) If $A(p,q,r)B^{qs}_r = C^s_p$, where B^{qs}_r is an arbitrary tensor and C^s_p a tensor, then prove that $A(p,q,r)$ is a tensor.

(c) Find the covariant and contravariant components of a tensor in cylindrical co-ordinates (ρ, ϕ, z) if its covariant components in rectangular co-ordinates are $xy, 2y - z^2, xz$.

2. (a) Define the following:

i. Christoffel symbols of first and second kind,

ii. Geodesics.

(b) Prove the following:

i. $[pq, r] = [qp, r]$,

ii. $\Gamma^s_{pq} = \Gamma^s_{qp}$,

iii. $[pq, r] = g_{rs}\,\Gamma^s_{pq}$.

and Corresponding Geodesic Equa

(c) Find the second kind of the Christoffel symbol in spherical co-ordinates (r, θ, ϕ).

3. (a) i. Explain the term "**Covariant derivative**" as applied to a tensor of type A^a_{bc}.

ii. Write the covariant derivative with respect to x^q of each of the following tensors.

$$A^{jk}_{l}, \quad A^{j}_{klm}, \quad A^{jk}_{lmn}.$$

(b) With the usual notations, prove the following:

i. $\dfrac{\partial g_{pq}}{\partial x^m} = [pm, q] + [qm, p]$;

ii. $\dfrac{\partial g^{pq}}{\partial x^m} = -\,g^{pn}\,\Gamma^q_{mn} - g^{qn}\,\Gamma^p_{mn}$.

Deduce that the covariant derivatives of the tensors g_{jk}, g^{jk}, δ^j_k are zero.

(c) Using the covariant derivative of metric tensor, prove that

$$\Gamma^e_{ca} = \frac{1}{2}\, g^{eb}[\partial_c(g_{ab}) + \partial_a(g_{cb}) - \partial_b(g_{ca})],$$

where $\partial_i = \dfrac{\partial}{\partial x^i}$.

SECTION B

NUMERICAL ANALYSIS

4. (a) Use induction on r to show that the Horner algorithm

$$k_0 = a_n,$$
$$k_r = b\, k_{r-1} + a_{n-1} \quad for \quad r = 1, 2, \ldots, n,$$

generates a sequence (k_r) that satisfies

$$k_r = a_n\, b^r + a_{n-1} b^{r-1} + \cdots + a_{n-r} \quad \text{for} \quad r = 0, 1, \ldots n.$$

Show also that

$$a_n\, x^n + a_{n-1}\, x^{n-1} + \ldots + a_0 = (x - b)(k_0\, x^{n-1} + k_1 x^{n-2} + \cdots + k_{n-1}) + k_n.$$

Hence find the following:

i. the quotient polynomial and remainder when $p(x) = 3x^5 + 5x^4 + 8x^2 + 7x + 4$ is devided by $x + 2$;

ii. the Taylor series of $p(x)$ about the point $x = -2$.

(b) Explain what is meant by

i. fixed point representation ,

ii. floating point representation.

For base 16, round the number $\dfrac{\pi}{2} = (1.921FB54\ldots)_{16}$ to

i. 5 places in fixed point ,

ii. 5 digits in floating point.

(In base 16, $A = 10$, $B = 11$, $C = 12$, $D = 13$, $E = 14$, $F = 15$.)

5. Let $f_0, f_1, \ldots, f_m$ be a sturm sequence of polynomial of decreasing degree with the properties

(a) $f_m(x) \neq 0$ for any x in $[a, b]$,

(b) if $f_r(\alpha) = 0 \quad (1 \le r < m - 1)$, then $f_{r-1}(\alpha) f_{r+1}(\alpha) < 0$,

(c) if $f_0(\alpha) = 0$, then $f_1(\alpha) f_0'(\alpha) > 0$: where $\alpha \in [a, b]$

Let $V(x)$ denote the number of sign charges in this sequence at any fixed value of x (for which $f_0(x) \neq 0$). Prove that f_0 has $V(a) - V(b)$ zeros in the interval $[a, b]$.

Construct a sturm sequence for the polynomial

$$P(x) = x^4 + 2x - 1 .$$

Show that

(a) only two roots of P are real;

(b) one root is positive and the other is negative;

(c) the positive root lies in the interval $[0, 1]$.

6. Define what is meant by the statement that a function $g : [a, b] \longrightarrow [a, b]$ is a contraction mapping on $[a, b]$.

Let g be a contraction mapping on $[a, b]$ with Lipschitz constant k. Show that there is a unique $c \in [a, b]$ such that $c - g(c) = 0$. Show also that c is the limit of the sequence $\{x_n\}$ given by

$$x_{n+1} = g(x_n), \quad n = 0, 1, \ldots, \quad \text{for any } x_0 \in [a, b].$$

Prove that $|x_1 - c| \leq \dfrac{k}{1 - k}|x_1 - x_0|$

Show that there exists a unique $c \in [0, 1]$ such that $c = g(x)$ where $g(x) = \dfrac{1}{2}e^{x-1}$. Use the above inequality to find lower and upper bound for c, taking $x_0 = 0.5$.