EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE - 2008/2009

SECOND SEMESTER (Sep./Oct.,2010)

MT 102 - ANALYSIS I (Real Analysis)

Answer all questions | Time: Three hours

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1. (a) i. Define the terms *Supremum* and *Infimum* of a non-empty subset of $\mathbb{R}$.

   ii. State the Completeness property of $\mathbb{R}$.

   Prove that every non-empty bounded below subset of $\mathbb{R}$ has infimum.

   (b) i. Prove that an upper bound u of a non-empty bounded above subset S of $\mathbb{R}$ is the supremum of S if and only if for every $\epsilon > 0$, there exists an $x \in S$ such that $x > u - \epsilon$.

   ii. Prove that $\sup(-S) = \inf(S)$ for any non-empty bounded subset S of $\mathbb{R}$.

   (c) Find the supremum and infimum of the set

$$S = \left\{ \frac{1}{2^m} + \frac{1}{3^n} : m, n \in \mathbb{N} \right\}.$$

2. Define what is meant by a *monotone sequence*.

   (a) Prove that every convergent sequence is bounded.

   Is the converse true? Justify your answer.

(b) i. Prove that $\lim_{n\to\infty} z^n = 0$, if $|z| < 1$.

ii. Let (x_n) be a sequence such that $x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$ for $n \geq 3$.

Using mathematical induction prove that

$$x_n = \frac{x_1}{3}\left[1 - \left(\frac{-1}{2}\right)^{n-2}\right] + \frac{x_2}{3}\left[2 + \left(\frac{-1}{2}\right)^{n-2}\right] \text{ for } n \geq 3.$$

Hence find the limit in terms of x_1 and x_2.

(c) State Monotone convergent theorem.

Prove that the sequence $\left\{\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}\right\}$ is convergent.

3. (a) Define the following terms:

i. a subsequence of a sequence,

ii. Cauchy sequence.

(b) State and prove the Bolzano-Weierstrass theorem.

(c) Prove that the sequence of real numbers is Cauchy if and only if it is convergent.

Hence show that the sequence (x_n) given by $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$ is not convergent.

4. (a) Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a function. Define what is meant by a limit of f at a point x_0 in A is l.

Prove that $\lim_{x\to 2}(x^2 + x - 1) = 5$.

(b) I. If $\lim_{x\to a} f(x) = l$, then show that $\lim_{x\to a} |f(x)| = |l|$.

Is the converse true?Justify your answer.

II. Let $f : \mathbb{R} \to \mathbb{R}$ be a function and $\lim_{x\to a} f(x) = l(\neq 0)$.

Prove the following:

i. there exists $\delta > 0$ such that $\frac{|l|}{2} < |f(x)| < \frac{3|l|}{2}$ for all x satisfying $0 < |x - a| < \delta$;

ii. $\lim_{x\to a} \frac{1}{f(x)} = \frac{1}{l}$, if $f(x) \neq 0$ for all $x \in \mathbb{R}$.

5. (a) Let $a \in A(\subseteq \mathbb{R})$ and let $f : A \to \mathbb{R}$, prove that f is not continuous at a if and only if there exist a sequence (x_n) in A that converges to a but the sequence

$f(x_n)$ does not converges to $f(a)$.

Hence prove that the function $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 1-x & \text{if } x \in \mathbb{Q}^C \end{cases}$$

is continuous only at $x = \dfrac{1}{2}$.

(b) Let $f : [a,b] \to \mathbb{R}$ be a continuous function on $[a,b]$. Prove that it is bounded on $[a,b]$.

Is the converse true?Justify your answer.

(c) Discuss the continuity of the following function at $x = 0$.

$$f(x) = \begin{cases} \dfrac{x\, e^{\frac{1}{x}}}{1+e^{\frac{1}{x}}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

(a) Suppose that f and g are continuous on $[a,b]$ differentiable on (a,b) and $g'(x) \neq 0$ for all $x \in (a,b)$. Prove that there exists $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

If $f(d) = g(d) = 0$ for some $d \in (a,b)$ deduce that

$$\lim_{x \to d} \frac{f(x)}{g(x)} = \lim_{x \to d} \frac{f'(x)}{g'(x)}.$$

(b) Evaluate the following limits:

i. $\displaystyle\lim_{x \to 0} \frac{e^x + e^{-x} - 2\cos x}{x \sin x}$;

ii. $\displaystyle\lim_{x \to 0} \frac{\sin x - x\cos x}{x^2 \sin x}$.