EASTERN UNIVERSITY, SRI LANKA
DEPARTMENT OF MATHEMATICS
FIRST EXAMINATION IN SCIENCE -2008/2009
SECOND SEMESTER (Oct./Nov., 2010)
MT 107 - THEORY OF SERIES
(PROPER & REPEAT)

Answer all Questions | Time: Two hours

1. (a) Define what is meant by the convergent or divergent of an infinite series of real numbers $\sum_{n=1}^{\infty} a_n$. Consider the series $\sum_{n=1}^{\infty} a_n$ whose n^{th} term is

$$\tan^{-1}\left(\frac{\frac{1}{n}-\frac{1}{(n+1)}}{1+\frac{1}{n(n+1)}}\right).$$

Show that the series $\sum_{n=1}^{\infty} a_n$ converges and find its sum.

(b) A necessary condition for a series $\sum_{n=1}^{\infty} a_n$ to converge is that $\lim_{n\to\infty} a_n = 0$.

Is it true that, it is a sufficient condition for the convergence of $\sum_{n=1}^{\infty} a_n$? Justify your answer.

(c) Prove that if $a_1 + a_2 + a_3 + \ldots$ converges to a, then

$$\frac{1}{2}(a_1+a_2)+\frac{1}{2}(a_2+a_3)+\frac{1}{2}(a_3+a_4)+\ldots$$

converges.

2. (a) Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two series of non-negative real numbers such that

   i. $a_n \leq kb_n$ for all $n \in \mathbb{N}$ and some positive real number k, and

   ii. $\sum_{n=1}^{\infty} b_n$ converges.

   Then show that $\sum_{n=1}^{\infty} a_n$ converges.

(b) i. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are real series of non-negative terms such that $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$, $n = 1, 2, \ldots$, and if $\sum_{n=1}^{\infty} b_n$ is convergent, then prove that $\sum_{n=1}^{\infty} a_n$ converges.

   ii. If $\sum_{n=1}^{\infty} a_n$ is a convergent series of non-negative terms, then show that $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

(c) i. If the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent and the sequence $\{b_n\}$ is bounded, then prove that $\sum_{n=1}^{\infty} a_n b_n$ is absolutely convergent.

   ii. If we replace absolutely convergent by convergent what happen to the result of the above part? Justify your answer.

3. (a) Let $\sum_{n=1}^{\infty} a_n$ be a given series with real valued terms and define

$$p_n = \frac{|a_n| + a_n}{2}, \quad q_n = \frac{|a_n| - a_n}{2}, \qquad n = 1, 2, \ldots.$$

   Then show that

   i. if $\sum_{n=1}^{\infty} a_n$ is conditionally convergent, then both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ diverge.

   ii. if $\sum_{n=1}^{\infty} |a_n|$ converges then both $\sum_{n=1}^{\infty} p_n$ and $\sum_{n=1}^{\infty} q_n$ converge and $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} p_n - \sum_{n=1}^{\infty} q_n$.

(b) Define the term *sequence of bounded variation.*

   i. Let $\sum_{n=1}^{\infty} a_n$ be a convergent real series and $\{b_n\}$ be a real sequence of bounded variation, then prove that $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

ii. In part (i), if the sequence $\{b_n\}$ of real numbers is bounded, what happens to the result of that part? Justify your answer.

iii. If $\sum_{n=1}^{\infty} a_n$ is a convergent real series and $\{b_n\}$ is a monotonic and bounded real sequence, then show that $\sum_{n=1}^{\infty} a_n b_n$ is convergent.

4. (a) Let $\sum_{n=1}^{\infty} z_n$ be a series of complex numbers.

i. Show that the geometric series $1 + z + z^2 + \ldots$ has the sum $\dfrac{1}{(1-z)}$ when $|z| < 1$.

ii. Show that $1 + \dfrac{z}{1+z} + \dfrac{z^2}{(1+z)^2} + \ldots$ converges when $z \in E = \{z/Re(z) > -\frac{1}{2}\}$.

Hence find the sum of the series.

(b) State the comparison test for series of complex numbers.

Hence check whether the series $\sum_{n=1}^{\infty} \dfrac{(n+i)(1+ni)}{n^2}$ converges or diverges.

(c) Let $f(z) = \ln(1+z)$ where the branch which has the value zero when $z = 0$ is considered.

i. Expand $f(z)$ in a Taylor series about $z = 0$.

ii. Determine the region of convergence for the series in part (i).

iii. Expand $\ln\left(\dfrac{1+z}{1-z}\right)$ in a Taylor series about $z = 0$.