



EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE - 2008/2009

SECOND SEMESTER (Sep./Nov., 2010)

ST 104 - DISTRIBUTION THEORY

(RE-REPEAT)

Answer all questions

Time : Three hours

1. (a) If U has a χ^2 distribution with n degrees of freedom,

$$\theta = \begin{cases} \frac{e^{-\frac{n}{2}} u^{\frac{(n-1)}{2}}}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})}, & u \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Find $E(U)$ and $V(U)$.

- (b) Let $Y_1, Y_2, Y_3, \dots, Y_n$ be random sample from a normal distribution with mean μ and variance σ^2 . Find the $E(S^2)$ and $V(S^2)$, where

$$S^2 = \frac{1}{n-1} \sum_{i=1}^n [Y_i - \bar{Y}]^2 \quad \text{and} \quad \bar{Y} = \frac{1}{n} \sum_{i=1}^n Y_i.$$

2. If X and Y are two random variables have density function

$$f_{XY}(x, y) = \begin{cases} \frac{1}{8}(6 - x - y), & \text{if } 0 < x < 2, 2 < y < 4; \\ 0, & \text{otherwise.} \end{cases}$$

Find

- marginal densities of X and Y .
 - joint cumulative distribution function.
 - $P(X < 1, Y < 3)$.
 - $P(X + Y < 3)$.
 - $P(X < 1 | Y < 3)$.
3. (a) A particular fast-food outlet is interested in the joint behavior of the random variables Y_1 , defined as the total time between a customer's arrival at the store and leaving the service window, and Y_2 , the time that a customer waits in line before reaching the service window. Because Y_1 contains the time a customer waits in line, we must have $Y_1 \geq Y_2$. The relative frequency distribution of observed values of Y_1 and Y_2 is modeled by the probability density function

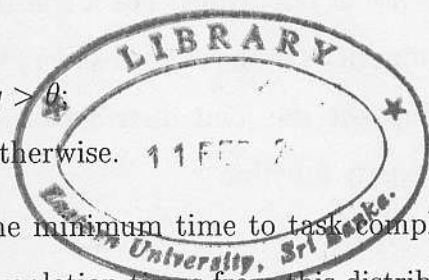
$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 < \infty; \\ 0, & \text{otherwise} \end{cases}$$

Another random variable of interest is $U = Y_1 - Y_2$, the time spent at the service window.

- Find the probability density function for U .
 - Find $E(U)$ and $V(U)$.
- (b) Let Y_1, Y_2, \dots, Y_n be independent uniformly distributed random variables on the interval $[0, \theta]$.
- Find the probability distribution function of $Y_{(n)} = \max(Y_1, Y_2, \dots, Y_n)$.
 - Find the density function of Y_n .
 - Suppose that the number of minutes that you need to wait for a bus uniformly distributed on the interval $[0, 15]$. If you take the bus five times, what is the probability that your longest wait is less than 10 minutes?

- (a) Suppose that the length of time Y that takes a worker to complete a certain task, has the probability density function

$$f(y) = \begin{cases} e^{-(y-\theta)}, & y > \theta; \\ 0, & \text{otherwise.} \end{cases}$$



where θ is a positive constant that represents the minimum time to task completion. Let Y_1, Y_2, \dots, Y_n denote the random sample of completion times from this distribution.

- i. Find the density function for $Y_{(1)} = \min(Y_1, Y_2, \dots, Y_n)$.
- ii. Find $E(Y_{(1)})$.

- (b) Let X be a standard normal variate. Show that $Y = X^2$ is a chi-square random variable with degrees of freedom 1.

- (c) Let Y_1, Y_2, \dots, Y_n be a random sample of size n from a normal distribution with a mean μ and a variance of σ^2 . If $Z_i = \frac{(Y_i - \mu)}{\sigma}$, show that $\sum_{i=1}^n Z_i^2 = \sum_{i=1}^n \left[\frac{(Y_i - \mu)}{\sigma} \right]^2$ is a χ^2 distribution with n degrees of freedom.

5. Let Y be a random variable with density function given by

$$f(y) = \begin{cases} \frac{3}{2}y^2, & -1 \leq y \leq 1; \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find the density function of $U_1 = 3Y$.
- (b) Find the density function of $U_2 = 3 - Y$.
- (c) Find the density function of $U_3 = Y^2$.
- (d) Find $V(U_1)$, $V(U_2)$ and $V(U_3)$.

6. A certain process for producing an industrial chemical yields a product containing types of impurities. For a specify sample from this process, let Y_1 denote the proportion of type I impurities in the sample and Y_2 the proportion of type II impurity among all impurities in the sample. Suppose the joint distribution of Y_1 and Y_2 can be modeled by the following probability density function

$$f(y_1, y_2) = \begin{cases} 2(1 - Y_1), & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1; \\ 0, & \text{otherwise.} \end{cases}$$

- (a) Find the probability density function of the proportion of type I impurities in the sample.
- (b) Find the expected value of the proportion of type II impurities in the sample.