EASTERN UNIVERSITY, SRI LANKA

DEPARTMENT OF MATHEMATICS

FIRST EXAMINATION IN SCIENCE - 2008/2009

SECOND SEMESTER (Sep./Nov., 2010)

ST 104 - DISTRIBUTION THEORY

(RE-REPEAT)

nswer all questions Time : Three hour

1. (a) If U has a χ^2 distribution with n degrees of freedom,

$$\theta = \begin{cases} \dfrac{e^{-\frac{n}{2}} u^{(\frac{n}{2}-1)}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}, & u \neq 0; \\ 0, & \text{otherwise.} \end{cases}$$

Find $E(U)$ and $V(U)$.

(b) Let $Y_1, Y_2, Y_3, ..., Y_n$ be random sample from a normal distribution with mean μ and variance σ^2. Find the $E(S^2)$ and $V(S^2)$, where

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}[Y_i - \overline{Y}]^2 \quad \text{and} \quad \overline{\mathrm{Y}} = \frac{1}{\mathrm{n}}\sum_{\mathrm{i=1}}^{\mathrm{n}} \mathrm{Y_i}.$$

2. If X and Y are two random variables have density function

$$f_{XY}(x,y) = \begin{cases} \frac{1}{8}(6-x-y), & \text{if } 0 < x < 2, 2 < y < 4; \\ 0, & \text{otherwise.} \end{cases}$$

Find

(a) marginal densities of X and Y.

(b) joint cumulative distribution function.

(c) $P(X < 1, Y < 3)$.

(d) $P(X + Y < 3)$.

(e) $P(X < 1 | Y < 3)$.

3. (a) A particular fast-food outlet is interested in the joint behavior of the random va Y_1, defined as the total time between a customer's arrival at the store and leavi service window, and Y_2, the time that a customer waits in line before reachi service window. Because Y_1 contains the time a customer waits in line, we mus $Y_1 \geq Y_2$. The relative frequency distribution of observed values of Y_1 and Y_2 modeled by the probability density function

$$f(y_1, y_2) = \begin{cases} e^{-y_1}, & 0 \leq y_2 \leq y_1 < \infty; \\ 0, & \text{otherwise} \end{cases}$$

Another random variable of interest is $U = Y_1 - Y_2$, the time spent at the window.

i. Find the probability density function for U.

ii. Find $E(U)$ and $V(U)$.

(b) Let $Y_1, Y_2, ..., Y_n$ be independent uniformly distributed random variables on the in $[0, \theta]$.

i. Find the probability distribution function of $Y_{(n)} = \max(Y_1, Y_2, ..., Y_n)$.

ii. Find the density function of Y_n.

iii. Suppose that the number of minutes that you need to wait for a bus unif distributed on the interval [0,15]. If you take the bus five times, what probability that your longest wait is less than 10 minutes?

(a) Suppose that the length of time Y that takes a worker to complete a certain task, has the probability density function

$$f(y) = \begin{cases} e^{-(y-\theta)}, & y > \theta; \\ 0, & \text{otherwise.} \end{cases}$$

where θ is a positive constant that represents the minimum time to task completion. Let $Y_1, Y_2, ..., Y_n$ denote the random sample of completion times from this distribution.

i. Find the density function for $Y_{(1)} = \min(Y_1, Y_2, ..., Y_n)$.

ii. Find $E(Y_{(1)})$.

(b) Let X be a standard normal variate. Show that $Y = X^2$ is a chi-square random variable with degrees of freedom 1.

(c) Let $Y_1, Y_2, ..., Y_n$ be a random sample of size n from a normal distribution with a mean μ and a variance of σ^2. If $Z_i = \dfrac{(Y_i - \mu)}{\sigma}$, show that $\sum_{i=1}^{n} Z_i^2 = \sum_{i=1}^{n} \left[\dfrac{(Y_i - \mu)}{\sigma}\right]^2$ is a χ^2 distribution with n degrees of freedom.

. Let Y be a random variable with density function given by

$$f(y) = \begin{cases} \frac{3}{2}y^2, & -1 \leq y \leq 1; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the density function of $U_1 = 3Y$.

(b) Find the density function of $U_2 = 3 - Y$.

(c) Find the density function of $U_3 = Y^2$.

(d) Find $V(U_1)$, $V(U_2)$ and $V(U_3)$.

6. A certain process for producing an industrial chemical yields a product containin types of impurities. For a specify sample from this process, let Y_1 denote the proport impurities in the sample and Y_2 the proportion of type I impurity among all impurities f Suppose the joint distribution of Y_1 and Y_2 can be modeled by the following proba density function

$$f(y_1, y_2) = \begin{cases} 2(1 - Y_1), & 0 \leq y_1 \leq 1, 0 \leq y_2 \leq 1; \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the probability density function of the proportion of type I impurities i sample.

(b) Find the expected value of the proportion of type II impurities in the sample.