EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE - 2001/2002

(April/May.'2002)

FIRST SEMESTER

MT 201 - VECTOR SPACES AND MATRICES

Answer all questions

Time : Three hours

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1. (a) Explain what is meant by

   i. a vector space;

   ii. a subspace of a vector space.

   (b) Let **V** be a vector space over a field **F** and **W** be a non-empty subset of **V**. Prove that **W** is a subspace of **V** if, and only if $ax + by \in \mathbf{W}$ for every $x, y \in \mathbf{W}$ and for every $a, b \in \mathbf{F}$.

   (c) Let $V = \{x \;/\; x \in \Re, \;\; x > 0\}$. Define addition and scalar multiplication as follows:

$$x \oplus y = xy \qquad \text{for} \quad x, y \in V,$$

$$r \odot x = x^r \qquad \text{for} \;\; r \in \Re, \quad x \in V.$$

   Show that $(V, \oplus, \odot)$ is a vector space over $\Re$.

(d) Which of the following sets are subspaces of $\Re^3$? In each case justify your answer.

i. $W_1 = \{(x, y, z) \in \Re^3 / x + y + z = 1\}$

ii. $W_3 = \{(x, y, z) \in \Re^3 / x + y^2 = 0\}$

2. (a) Define the following terms:

i. A linearly independent set of vectors;

ii. A basis for a vector space.

(b) Prove that the non-zero vectors $v_1, v_2, \cdots, v_n$ of a vector space V over the field F are linearly dependent if and only if one of them say $v_i (2 \leq i \leq n)$ is a linear combination of the preceding vectors.

(c) i. State the Dimension Theorem.

ii. Let $U = < \{(1, 1, 0, -1), (1, 2, 3, 0), (2, 3, 3, -1)\} >$

$W = < \{(1, 2, 2, -2), (2, 3, 2, -3), (1, 3, 4, -3)\} >$.

Find

A. $\dim(U + W)$;

B. $\dim(U \cap W)$.

3. (a) Define

i. Range space $R(T)$;

ii. Null space $N(T)$;

of a linear transformation T from a vector space V into another vector space W.

Let T be a linear transformation from a finite dimensional vector space V into a finite dimensional vector space W. Prove that the image of any linearly independent subset of V is a linearly independent subset of W if, and only if $N(T) = \{0\}$.

(b) Find $R(T)$ and $N(T)$ of the linear transformation $T : \Re^3 \longrightarrow \Re^3$ defined by

$$T(x, y, z) = (x - y + 2z, \quad 2x + y, \quad -x - 2y + 2z).$$

Verify the equation dim V= dim $N(T)$ + dim $R(T)$ for the above linear transformation.

(c) The linear transformation $T : \Re^3 \longrightarrow \Re^2$ is defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, \quad x_1 - x_3).$$

In $\Re^2$, $B_1 = \{(1,1), (1,-1)\}$ is a basis, and in $\Re^3$, $B_2 = \{(1,1,0), (0,1,1), (1,0,1)\}$ is a basis. Obtain

i. the matrix of T with respect to the standard basis of $\Re^3$ and the basis B_1 of $\Re^2$.

ii. the matrix of T with respect to the basis B_2 of $\Re^3$ and the standard basis of $\Re^2$.

iii. the matrix of T with respect to the basis B_2 of $\Re^3$ and the basis of $\Re^2$.

4. (a) Define the following terms as applied to an $n \times n$ matrix $A = (a_{ij})$

i. Row space,

ii. Echelon form,

iii. Row reduced echelon form.

(b) Let A be an $n \times n$ matrix. Prove that,

i. row rank of A is equal to column rank of A ;

ii. if B is an $n \times n$ matrix, obtained by performing an elementary row operation on A, then $r(A) = r(B)$.

(c) Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & -2 \end{pmatrix}.$$

(d) Find the row reduced echelon form of the matrix

$$\begin{bmatrix} 5 & 6 & 8 & -1 \\ 4 & 3 & 0 & 0 \\ 10 & 12 & 16 & -2 \\ 1 & 2 & 0 & 0 \end{bmatrix}.$$

5. (a) Define the the following terms as applied to an $n \times n$ matrix $A = (a_{ij})$.

i. Cofactor A_{ij} of an element a_{ij},

ii. Adjoint of A.

Prove that

$$A \cdot (\text{adj}A) = (\text{adj}A) \cdot A = \det A \cdot I$$

where I is the $n \times n$ identity matrix.

(b) If A and B are two $n \times n$ non-singular matrices, then prove that

i. $\text{adj}(\alpha A) = \alpha^{n-1} \cdot \text{adj}A$ for every real number α,

ii. $\text{adj}(AB) = (\text{adj}B)(\text{adj}A)$,

iii. $\text{adj}(A^{-1}) = (\text{adj}A)^{-1}$,

iv. $\text{adj}(\text{adj}A) = (\det A)^{n-2}A$,

v. $\text{adj}(\text{adj}(\text{adj}A)) = (\det A)^{n^2-3n+3}A^{-1}$.

(c) Find the inverse of the matrix

$$\begin{bmatrix} 3 & 4 & 5 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{bmatrix}.$$

6. (a) State the Necessary and Sufficient condition for a system of linear equations to be consistent.

The system of equations

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 5, \\ 3x_1 + 2x_2 - 4x_3 + 7x_4 &= k + 4, \\ x_1 + x_2 - x_3 + 2x_4 &= k - 1; \end{aligned}$$

is known to be consistent. Find the value of k and the general solution of the system.

(b) State Cramer's rule and use it to solve the following system of linear equations.

$$\begin{aligned} x + 2y + 3z &= 10, \\ 2x - 3y + z &= 1, \\ 3x + y - 2z &= 9. \end{aligned}$$