EASTERN UNIVERSITY, SRI LANKA

SPECIAL REPEAT EXAMINATION IN SCIENCE - 2007/2008

THIRD YEAR, FIRST AND SECOND SEMESTER(FEB.,2010)

MT 301 - ALGEBRA III [GROUP THEORY]

Answer all questions **Time: Three hours**

1. Define the terms **group** and **subgroup**.

   (a) Prove that in a finite group G, the order of each element divides the order of G. Hence prove that $x^{|G|} = e$, $\forall\, x \in G$, where e is the identity element of G.

   (b) Let H be a subgroup of a group G. Prove that $H^{-1} = H$.
   Is it true that, if $H^{-1} = H$, then H is a subgroup of G? Justify your answer.

   (c) i. Let G be a group of order 27. Prove that G contains a subgroup of order 3.

   ii. Let H and K be different subgroups of G, each of order 16. Prove that $24 \leq |H \cup K| \leq 31$.

2. Define the terms **cyclic group** and **abelian group**.

   (a) Prove that

   i. Every subgroup of a cyclic group is cyclic.
   Is the converse true? Justify your answer.

   ii. Every cyclic group is abelian.

   (b) If G is an infinite cyclic group generated by an element $a \in G$, then show that the powers of a are distinct.

(c) Let H be a subgroup of a group G, and let $a, b \in G$.

Prove the following:

i. If $Ha \subseteq Hb$, then $Ha = Hb$;

ii. If $Ha \cap Hb \neq \phi$, then $Ha = Hb$;

iii. $Ha = Hb$ if and only if $ab^{-1} \in H$.

3. Define **normal subgroup** of a group, **homomorphism** and **isomorphism**.

(a) Let $\phi : G \to G_1$ be a homomorphism. If $H \leq G$, then prove that $\phi(H) \leq G_1$. Further, if $H \trianglelefteq G$, and ϕ is onto, then show that $\phi(H) \trianglelefteq G_1$.

(b) i. Let G_1 be a group of all real 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $ad - bc \neq 0$ under matrix multiplication and G_2 be a group of all non-zero real numbers under multiplication. If $\phi : G_1 \to G_2$ defined by

$$\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc,$$

show that ϕ is a homomorphism.

ii. Given that G is the group of all positive real numbers under multiplication and G' be the group of all real numbers under addition. Let $f : G \to G'$ be defined by

$f(x) = \log(x),\ \forall\, x \in G$. Show that f is an isomorphism.

4. (a) State and prove the first isomorphism theorem.

Let H and K be two normal subgroups of a group of G, such that $K \subseteq H$. Prove the following:

i. $K \trianglelefteq H$;

ii. $H/K \trianglelefteq G/K$;

iii. $\dfrac{G/K}{H/K} \cong G/H$.

(b) What is meant by an index of a subgroup of a group G.

Let H and K be two subgroups of a finite group G and $K \subseteq H$. Prove that $[G : K] = [G : H][H : K]$, where $[G : K]$ is the index of K in G.

5. (a) Define the term $p-$group.

   i. Prove that homomorphic image of a $p-$group is a $p-$group.

   ii. Let G be a finite abelian group and p be a prime number such that p is a divisor of the order of G. Prove that G has an element of order p.

   (b) Let H and K be two subgroups of a group G. Prove that G is the direct sum of H and K if and only if

   i. each $x \in G$ can be uniquely expressed in the form $x = hk$, where $h \in H, k \in K$.

   ii. $hk = kh, \forall\ h \in H, k \in K$.

6. Define **permutation on n symbols, cycle of order r** and **transposition** as applied to a permutation group.

   (a) Prove that the permutation group on n symbols S_n is a finite group of order $n!$. Is S_n abelian for $n > 2$? Justify your answer.

   (b) Prove that every permutation in S_n can be expressed as a product of transpositions. Hence show that an even permutation can be expressed as a product of even number of transpositions.

   Express the following permutation as a product of transpositions and hence determine whether it is odd or even.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}$$

   (c) Prove that the set of even permutations forms a normal subgroup of S_n. Hence show that S_n/A_n is a cyclic group of order 2.