EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE 2001/2002

(April/May'2002)

FIRST SEMESTER

MT 203 - EIGENSPACE & QUADRATIC FORMS

Answer all questions

Time : Two hours

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1. Define the term "an eigenvalue" of a linear transformation.

   Explain what is meant by "a linear transformation is diagonalizable."

   (a) Prove that eigenvectors that corresponding to distinct eigenvalues of a linear transformation $T : V \longrightarrow V$ are linearly independent, where V is a vector space.

   (b) Prove that if T is a linear transformation such that $T^2 = I$ then the sum of all eigenvalues of T is an integer.

   Find the eigenvalues for the linear transformation $T : \Re^3 \longrightarrow \Re^3$ such that $T(x, y, z) = (x + 2y + 2z,\ x + 2y - z,\ -x + y + 4z)$ where $x, y, z \in \Re$.

   Further find a non-singular matrix P such that $P^{-1}AP$ is diagonal, where A is the matrix representation of T.

2. (a) Define the term "skew- symmetric" as applied to an $n \times n$ matrix.

   Let A be a real skew-symmetric matrix with eigenvalue λ.

   i. Prove that λ is zero or purely imaginary, and $\overline{\lambda}$ is also an eigenvalue of A.

   ii. If $(A - \lambda I)^2 z = 0$ and $y = (A - \lambda I)z$ then by evaluating $(\overline{y})^t y$, show that $y = 0$, where y and z are n-column vectors.

   (b) Find an orthogonal transformation which reduces the following quadratic form to a diagonal form

$$2x_1^2 + 5x_2^2 + 2x_3^2 + 4x_1x_2 + 4x_2x_3 + 2x_1x_3.$$

3. Let λ_1 and λ_2 be two distinct roots of the equation $| A - \lambda B |= 0$, where A and B are real symmetric matrices and let u_1 and u_2 be two vectors satisfying $(A - \lambda_i B)u_i = 0$ for $i = 1, 2$. Prove that $u_1^T B u_2 = 0$.

   Simultaneously reduce the following pair of quadratic forms

$$\begin{aligned} \phi_1 &= x_1^2 + x_2^2 + x_3^2 + 2x_2x_3 - 2x_1x_3 - 2x_1x_2 \\ \phi_2 &= 3x_1^2 + x_2^2 + 3x_3^2 - 2x_2x_3 - 2x_1x_3 + 2x_1x_2 \end{aligned}$$

4. What is meant by an "inner product" on a vector space?

   (a) Prove that, for any vectors x, y in an inner product space,

   $$|< x, y >| \leq \| x \| \| y \| .$$

   (b) State Gram-Schmidt process and use it to find the orthonormal set for span of S in $\Re^3$, where $S = \{(1,1,1),(0,1,1),(0,0,1)\}$.

   (c) Let X be an inner product space and M be a finite dimensional subspace of X. Prove that $X = M \oplus M^{\perp}$, where $M^{\perp}$ is orthogonal complement of M and $\oplus$ denotes the direct sum.