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# EASTERN UNIVERSITY, SRI LANKA

## SECOND EXAMINATION IN SCIENCE 2003/2004

## Nov./Dec. 2004

## FIRST SEMESTER

## MT 201 - VECTOR SPACES AND MATRICES

**Answer all questions**

**Time: 3 hours**

1. (a) Explain what is meant by a vector space.

   (b) Let $V = \{f / f : \mathbb{R} \longrightarrow \mathbb{R}, f(x) > 0 \ \forall x \in \mathbb{R}\}$. For any $f, g \in$ V and for any $\alpha \in \mathbb{R}$ define an addition $\oplus$ and a scalar multiplication $\odot$ as follows:

$$(f \oplus g)(x) = f(x).g(x) \qquad \forall x \in \mathbb{R}$$

   and

$$(\alpha \odot f)(x) = \{f(x)\}^{\alpha} \qquad \forall x \in \mathbb{R}.$$

   Prove that $(V, \oplus, \odot)$ is a vector space over the set of real numbers $\mathbb{R}$.

   (c) An attempt is made to turn the set $\mathbb{Z}^2$ of pairs of integers into a vector space over the field $\mathbb{R}$ by defining:

$$(u, v) + (u', v') = (u + u', v + v')$$

$$\alpha(u, v) = (\lfloor \alpha \rfloor u, \lfloor \alpha \rfloor v)$$

   where $\lfloor \alpha \rfloor$ is the integer part of $\alpha$ and $u, u', v, v' \in \mathbb{Z}$. Is this is a vector space?

   Justify your answer.

2. (a) Define the followings:

   i. a linear independent set of vectors;

   ii. a basis for a vector space;

   iii. direct sum of two subspaces $W_1$ and $W_2$ of a vector space $V$.

(b) Let $S, W$ be two subspaces of a vector space $V$ over the field $\mathbb{F}$. Prove that; $V$ is a direct sum of $S$ and $W$ iff each vector $v \in V$ has a unique representation $v = s + w$ for some $s \in S, w \in W$.

Let $U$ and $W$ be two subspaces of $\mathbb{R}^3$ defined by

$U = \{(a,b,c)/a = b = c,\ a,b,c \in \mathbb{R}\}$ and $W = \{(0,p,q)\ /p,q \in \mathbb{R}\}$.

Show that; $\mathbb{R}^3 = U \oplus W$.

(c) i. Let $S$ be any non-empty linearly independent subset of a vector space $V$ over the field $\mathbb{F}$. What is meant by saying that "$S$ spans $V$".

Prove that; for any $v \in V$ the set $S \cup \{v\}$ is linearly independent iff $v \notin \langle S \rangle$.

ii. Prove that; any linearly independent subset of a finite dimensional vector space $V$ can be extended to a basis of $V$.

3. (a) State and prove the Dimension theorem for two subspaces of a finite dimensional vector space.

(b) Let $V$ be a finite dimensional vector space and $W$ be a subspace of $V$. Prove that; the quotient space $V/W$ is also finite dimensional and $\dim(\mathrm{V/W}) = \dim\mathrm{V} - \dim\mathrm{W}$.

(c) If $W_1 = \langle\{(1,0,2),(1,2,2)\}\rangle$ and $W_2 = \langle\{(1,1,0),(0,1,1\}\rangle$ are subspaces of $\mathbb{R}^3$. Find

   i. $\dim(\mathrm{W_1 + W_2})$;

   ii. $\dim(\mathrm{W_1 \cap W_2})$;

and verify that

$$\dim(\mathrm{W1 + W2}) = \dim\mathrm{W1} + \dim\mathrm{W2} - \dim(\mathrm{W_1 \cap W_2}).$$

4. (a) Let $T$ be a linear transformation from a vector space $V$ in to another vector space $W$. Define

   i. range space $R(T)$ and

   ii. null space $N(T)$.

   (b) Find $R(T), N(T)$ of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by $T(x, y, z) = (x + 2y - z, y + z, x + y - 2z), \quad \forall (x, y, z) \in \mathbb{R}^3$. Verify the equation

$$\dim(\mathbb{R}^3) = \dim(\mathrm{R(T)}) + \dim(\mathrm{N(T)})$$

   (c) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation defined by

$$T(x, y) = (x + 2y, 2x - y, -x).$$

   i. Find the matrix representation of $T$ with respect to the standard basis of $\mathbb{R}^2$ and $\mathbb{R}^3$.

   ii. Let $B_1 = \{(0, 1), (1, 1)\}$ be a basis of $\mathbb{R}^2$ and $B_2 = \{(0, 0, 1), (0, 1, 1), (1, 1, 1$ be a basis of $\mathbb{R}^3$. Find the matrix representation of $T$ with respect to the basis $B_1$ and $B_2$.

5. Define the term "non-singular" matrix.

   Let $J$ be the $n \times n$ real matrix with every entry equal to 1, so that $J^2 = nJ$, and let $A = \alpha I_n + \beta J$, where $\alpha, \beta$ are real numbers.

   (a) Show that $det A = \alpha^{n-1}(\alpha + n\beta)$.

   (b) If $\alpha \neq 0$ and $\alpha \neq -n\beta$ , prove that $A$ is non-singular by finding an inverse for it of the form $\frac{1}{\alpha}(I_n + \gamma J)$.

   Determine the inverse of the matrix

$$\begin{pmatrix} 7 & 5 & 5 & 5 & 5 \\ 5 & 7 & 5 & 5 & 5 \\ 5 & 5 & 7 & 5 & 5 \\ 5 & 5 & 5 & 7 & 5 \\ 5 & 5 & 5 & 5 & 7 \end{pmatrix}.$$

6. (a) State the necessary and sufficient condition for a linear equations to be consistent.

Find the condition which must be satisfied by $y_1, , y_2, y_3, y_4$ in order that the equations

$$x_1 - x_3 + 3x_4 + x_5 = y_1$$

$$2x_1 + x_2 - 2x_4 - x_5 = y_2$$

$$x_1 + 2x_2 + 2x_3 + 4x_5 = y_3$$

$$x_2 + x_3 + 5x_4 + 6x_5 = y_4$$

shall have a solution $x_1; x_2, x_3, x_4, x_5$. Find all the solutions for

$$y_1 = -3,\ y_2 = 5,\ y_3 = 6,\ y_4 = -2.$$

(b) State and prove Crammer's rule for $3 \times 3$ matrix and use it to solve;

$$2x - 5y + 2z = 7$$

$$x + 2y - 4z = 3$$

$$3x - 4y - 6z = 5.$$