EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE 2003/2004

FIRST SEMESTER

Oct/Nov 2004

ST 201 - STATISTICAL INFERENCE - 1

Answer all questions

Time : Two hours

1. (a) State and prove the Cramer-Rao inequality.

   (b) Given the probability density function,

   $$f(x,\theta) = [\pi\{1+(x-\theta)^2\}]^{-1} \quad ; \quad -\infty < x < \infty, \quad -\infty < \theta < \infty.$$

   Show that the Cramer- Rao lower bound of variance of an unbiased estimator of θ is $\frac{2}{n}$, where n is the size of the random sample from this distribution.

2. A particular type of a component is tested repeatedly until it fails. X, the number of tests until it fails, is found to have a geometric distribution with probability mass function

   $P(X = x) = p(1-p)^{x-1}$ $(x = 1, 2, \cdots)$, where p is a parameter. A random sample of n components is tested and the observed numbers of tests until failure are $X_1, X_2, \cdots, X_n$

   (a) Using moment generating function, or otherwise, show that $Y = \sum_{i=1}^{n} X_i$ has a negative binomial distribution with probability function

   $$P(Y = y) = \binom{y-1}{n-1} p^n(1-p)^{y-n} \quad ; \quad y = n, n+1, \cdots, \text{ where } n \text{ and } p$$

   are parameters.

(b) Show that Y is sufficient statistic for p.

(c) For $i = 1, 2, \cdots, n$, define the random variable V_i as follows:

$$V_i \begin{cases} 1, & \text{if } X_i = 1, \\ 0, & \text{if } X_i > 1. \end{cases}$$

Show that $\frac{(\sum V_i)}{n}$ is an unbiased estimator of p and find its variance.

(d) State the Rao-Block well theorem. By considering the conditional distribution of V_1 given Y, use this theorem to find $\hat{p}$, the unbiased estimator of p based on Y.

3. (a) Define

i. method of moment estimator.

ii. maximum likelihood estimator.

A random sample of n observations $X_1, X_2, \cdots, X_n$ is taken on a random variable X which has a normal distribution with mean μ and variance σ^2. Assuming σ^2 is known, find

i. The method of moment estimator for μ.

ii. The maximum likelihood estimator for μ.

(b) A random sample $X_1, X_2, \cdots, X_n$ is taken from a Poisson distribution with mean λ and it is required to estimate $\theta = \lambda^2$.

i. Show that the sample mean, $\bar{X}$, is a sufficient statistic for θ.

ii. Evaluate $E(\bar{X})$ and $E(\bar{X}^2)$ and hence find an unbiased estimator of θ based on $\bar{X}$.

iii. Find the Cramer- Rao lower bound for the variance of unbiased estimators of θ.

4. (a) A factory operates with two machines of type A and one machine of type B independently. The weekly repair costs Y for type A machines are normally distributed with mean μ_1 and variance σ^2. The weekly repair costs X for machines of type B are also normally distributed but with mean μ_2 and variance $3\sigma^2$. The expected repair cost per week for the factory is then $2\mu_1 + \mu_2$. If you are given a random sample $Y_1, Y_2, \cdots, Y_n$ on costs of type A machines and an independent random sample $X_1, X_2, \cdots, X_m$ on costs for type B machines, show how you would construct a 95% confidence interval for $2\mu_1 + \mu_2$. (Assume σ^2 is not known.)

(b) A random sample of $n_1 = 10$ observations on breaking strength of a type of glass gave $s_1^2 = 2.31$ (measurements were made in pounds per square inch). An independent random sample of $n_2 = 16$ measurements on a second machine, but with the same kind of glass gave $s_2^2 = 3.68$. Estimate the true variance ratio, $\frac{\sigma_1^2}{\sigma_2^2}$ in 90% confidence interval.

(c) Two brands of refrigerators, denoted by A and B, are each guaranteed for one year. In a random sample of 50 refrigerators of brand A, 12 were observed to fail before the guarantee period ended. A random sample of 60 brand B refrigerators also revealed 12 failures during the guarantee period. Estimate the true difference between proportions of failures during the guarantee period, $(p_A - p_B)$, with confidence coefficient 0.98.