EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE - 2003/2004

(Nov./Dec.'2004)

(FIRST SEMESTER)

MT 203 -EIGENSPACES & QUADRATIC FORMS

Answer all questions

Time:Two hours

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1. (a) Define the followings :

   i. eigenvalue;

   ii. characteristic polynomial of a square matrix;

   iii. algebraic multiplicity.

   (b) Let A be a non singular matrix in $\mathbb{R}_{n\times n}$. Show that the characteristic polynomial of A^{-1} is

   $$\chi_{A^{-1}}(t) = \frac{(-t)^n}{\det A}\chi_A\left(\frac{1}{t}\right), \qquad (t \neq 0).$$

   Deduce that if $\alpha_1, \alpha_2, ..., \alpha_n$ are the eigenvalues of A with algebraic multiplicities 1 then $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, ..., \frac{1}{\alpha_n}$ are the eigenvalues of A^{-1} with algebraic multiplicities 1.

   (c) i. Prove that eigenvectors that corresponding to distinct eigenvalues of a linear transformation $T : V \longrightarrow V$ are linearly independent, where V is a vector space.

   ii. Find all eigenvalues and a basis of each eigenspace of an operator $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (2x+y, y-z, 2y+4z)$ where $x, y, z \in \mathbb{R}$.

2. (a) Define the following terms for a square matrix:

   i. minimum polynomial;

   ii. irreducible polynomial.

   (b) Prove the followings:

   i. The characteristic polynomial of an $n \times n$ matrix A divides the n^{th} power of its minimum polynomial.

   ii. The characteristic polynomial and the minimum polynomial of an $n \times n$ matrix A have the same irreducible factors.

   (c) Let $M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ be a block diagonal matrix, where A and B are square matrices. Show that, the minimum polynomial $m(t)$ of M is the least common multiple of the minimum polynomials $g(t)$ and $h(t)$ of A and B respectively.

   Hence find the minimum polynomial of M given by

$$\begin{pmatrix} 2 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

3. (a) Find an orthogonal transformation which reduces the following quadratic form to a diagonal form.

$$5x_1^2 + 6x_2^2 + 7x_3^2 - 4x_1x_2 + 4x_2x_3.$$

   (b) Simultaneously diagonalize the following pair of quadratic forms

$$x_1^2 + 2x_2^2 + 8x_2x_3 + 12x_1x_2 + 12x_1x_3$$

$$3x_1^2 + 2x_2^2 + 5x_3^2 - 2x_1x_3 + 2x_2x_3.$$

4. Define the term "inner product space".

   (a) Prove that for any vectors x,y in an inner product space

   $$|\langle x, y \rangle| \leq \| x \| \| y \|.$$

   (b) Define the orthogonal complement, $S^{\perp}$ of a subset S in an inner product space V.

      i. Prove that in an inner product space the non-zero mutually orthogonal set of vectors are linearly independent.

      ii. Prove that $S \subseteq S^{\perp\perp}$ and that $S^{\perp\perp} = S$, when V has finite dimension. [state any result you may use]

   (c) State Gram-Schmidt process and use it to find the orthonormal basis of the subspace W of $\mathbb{C}^3$ spanned by $V_1 = (1, i, 0)$ and $V_2 = (1, 2, 1 - i)$.