EASTERN UNIVERSITY, SRI LANKA

SECOND EXAMINATION IN SCIENCE 2003/2004

(JUNE/JULY' 2005)

(Proper & Repeat)

SECOND SEMESTER

MT 202 - METRIC SPACE

Answer all questions | Time: Two hours

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1. Define the term *complete metric space.*

   (a) Let $C_{[0,1]}$ be the set of all continuous real valued functions on $[0,1]$.
   Define $d : C_{[0,1]} \times C_{[0,1]} \to \Re$ by
   $$d(x,y) = \int_0^1 \mid f(t) - g(t) \mid dt, \qquad \text{for all } f, g \in C_{[0,1]}.$$
   Prove that $(C_{[0,1]}, d)$ is a metric space and that is not complete.

   (b) Prove that a closed subspace of a complete metric space is complete.

2. (a) Let (X, d) be a metric space. Prove the following:

   i. $\mid d(x,z) - d(y,z) \mid \ \leq d(x,y)$, for all $x, y, z \in X$,

   ii. For any $x, y \in X$, $M_{(x,y)}$ is open;
   where $M_{(x,y)} = \{a \in X : d(x,a) > d(y,a)\}$

   (b) Let A be a subset of a metric space (X, d). Define the term *Frontier* ($\text{Fr}(A)$) of A.

   Prove that:

   i. $\text{ext}(A) = (\overline{A})^C$, where $\text{ext}(A) = (A^C)^o$,

ii. $\mathrm{Fr}(A) = \overline{A} \cap \overline{A^C}$,

iii. A is closed if and only if $\mathrm{Fr}(A) \subseteq A$,

iv. A is open if and only if $\mathrm{Fr}(A) \subseteq A^C$.

3. Define the term *compact set* in a metric space.

(a) Show that; $[a, b]$ is a compact subset of $\Re$ with respect to the usual metric,

(b) Let A be a compact subset of a metric space (X, d) and let $a \in X - A$. Prove that there exist open sets G and H such that $a \in G$, $A \subseteq H$ and $G \cap H = \Phi$. Hence, show that any compact subset of X is closed.

4. Let f be a function from a metric space (X, d_1) to a metric space (Y, d_2). Prove that the following statements are equivalent:

(a) the inverse image of every closed set contained in Y is closed in X,

(b) the inverse image of every open set contained in Y is open in X,

(c) f is continuous,

(d) $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for every subset B of Y.